Potpourri

Warm-Up Solutions

Math Circle Competition Team

August 27th, 2017

1. **(2007 AMC 12B)** Each face of a regular tetrahedron is painted either red, white, or blue. Two colorings are considered indistinguishable if two congruent tetrahedra with those colorings can be rotated so that their appearances are identical. How many distinguishable colorings are possible?

   $\boxed{15}$ A tetrahedron has 4 sides. The ratio of the number of faces with each color must be one of $4:0:0$, $3:1:0$, $2:2:0$, or $2:1:1$. The first ratio yields 3 appearances, one of each color. The second ratio yields $3 \cdot 2 = 6$ appearances, three choices for the first color, and two choices for the second. The third ratio yields $\binom{3}{2} = 3$ appearances since the two colors are interchangeable. The fourth ratio yields 3 appearances. There are three choices for the first color, and since the second two colors are interchangeable, there is only one distinguishable pair that fits them. The total is $3 + 6 + 3 + 3 = \boxed{15}$ appearances.

2. **(2002 AMC 10B)** Using the letters A, M, O, S, and U, we can form five-letter "words". If these "words" are arranged in alphabetical order, then the "word" $USAMO$ occupies which position?

   $\boxed{115}$ There are $4! \cdot 4$ "words" beginning with each of the first four letters alphabetically. From there, there are $3! \cdot 3$ with U as the first letter and each of the first three letters alphabetically. After that, the next "word" is $USAMO$, hence our answer is $4 \cdot 4! + 3 \cdot 3! + 1 = \boxed{115}$.

3. **(2003 AMC 10A)** A point (x, y) is randomly picked from inside the rectangle with vertices $(0,0), (4,0), (4,1)$, and $(0,1)$. What is the probability that $x < y$?

   $\boxed{1/8}$ The rectangle has a width of 4 and a height of 1. The area of this rectangle is $4 \cdot 1 = 4$. The line $x = y$ intersects the rectangle at $(0,0)$ and $(1,1)$. The area for which $x < y$ is the right isosceles triangle with side length 1 that has vertices at $(0,0)$, $(1,1)$, and $(0,1)$. The area of this triangle is $\frac{1}{2} \cdot 1^2 = \frac{1}{2}$. Therefore, the probability that $x < y$ is $\frac{\frac{1}{2}}{4} = \boxed{\frac{1}{8}}$.

Potpourri

In-Class Problems

Math Circle Competition Team

August 27th, 2017

1. The average of five distinct positive integers is 85, and the average of the three largest of the integers is 100. Compute the largest possible value of the second-smallest integer.

   $\boxed{98}$ The five numbers add to $5 \cdot 85 = 425$ while the three largest add to $3 \cdot 100 = 300$. Now since these are all different, the smallest of the three is less than 100. Since the second-smallest number is the same as the fourth largest, it must be even smaller - it must be less than 99. Could it be 98? Indeed, if we make the numbers 101, 100, 99, 98, and 27 all the conditions are satisfied, so the answer is $\boxed{98}$.

2. At Pascal Prep, 10% of the students take calculus and 15% of the students wear flip-flops. After some study, it turns out that 40% of the students who wear flip-flops take calculus. If a given student takes calculus, compute the probability that they wear flip-flops.

   $\boxed{0.6}$ The simplest way to do this is to just pick a convenient number of students at Pascal Prep. Let's say there are 100 students there. Then 10 take calculus, and 15 wear flip-flops. Of those 15, 40% (that is, 6) take calculus. Thus six of the ten of the students who take calculus wear flip-flops. So given that a student takes calculus, the probability they wear flip-flops is $\boxed{0.6}$ or 60%.

3. Compute the sum of all real values of x such that $(4^{x-1})^{x-3} = 8^x$.

   $\boxed{11/2}$ We have $4^{(x-1)(x-3)} = (2^3)^x$ so that $2^{2(x-1)(x-3)} = 2^{3x}$. Thus $2(x-1)(x-3) = 3x$. Moving all terms to one side, we receive $2x^2 - 11x + 6 = 0$. By Vieta's formulas, the sum of the roots of $ax^2 + bx + c$ is $-b/a$, so our answer is $\boxed{11/2}$.

4. Compute the least positive integer n such that $n!$ is divisible by 2015^2.

   $\boxed{62}$ First, we need to know the factors of 2015^2. Since $2015 = 5 \cdot 13 \cdot 31$, 2015^2 has two factors of 5, two of 13, and two of 31. The first time $n!$ has all these factors is when n reaches the second multiple of 31. Thus $n = \boxed{62}$.

5. If $1 + r + r^2 + \cdots = 17$, compute $1 + 2r + 3r^2 + 4r^3 + \cdots$.

   $\boxed{289}$ Note that the infinite geometric series $1 + r + r^2 + \cdots$ adds to $\frac{1}{1-r}$, so $\frac{1}{1-r} = 17$. For this series we let $S = 1 + 2r + 3r^2 + 4r^3 + \cdots$ and multiply both sides by r to receive $Sr = r + 2r^2 + 3r^3 + 4r^4 + \cdots$. Then subtracting the first from the second yields $S - Sr = S(1-r) = 1 + r + r^2 + r^3 + \cdots = 17$. Thus $S = 17 \cdot \frac{1}{1-r} = 17^2 = \boxed{289}$.