Algebra (and other) Conventions and Notations

Math Circle Competition Team

Number Sets

- $\mathbb{N} := \{1, 2, 3, 4, 5, ...\}$ — The Natural Numbers (Positive Integers)
- $\mathbb{N}_0 := \{0, 1, 2, 3, 4, ...\}$ — The Whole Numbers (Non-negative Integers)
- $\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$ — The Integers
- $\mathbb{Q} := \{..., -3, -\frac{3}{2}, -\frac{1}{7}, 0, 1, 2.5, \frac{100}{7}, ...\}$ — The Rational Numbers
- $\mathbb{R} := \{..., -\pi, -2, -\sqrt{3}, -\frac{1}{7}, 0, 1.5, e, \pi...\}$ — The Real Numbers
- $\mathbb{C} := \{..., -\pi, -3i, -\sqrt{17}, -\frac{i}{7}, 0, 5 + \frac{i}{2}, e^{i\pi}, \pi...\}$ — The Complex Numbers

Preliminary Set Notation

- $a \in A$ — "a is an element of set A"
- $\exists$ — "There exists..."
- $\forall$ — "For all..."
- $\mathbb{Z}_2$ — "The set of ordered integer pairs"
- $\mathbb{Z}_n$ — "The set of ordered integer n-tuples"
- $\mathbb{R}[X]$ — "The set of polynomials with real coefficients"

Functions

- f — "A function f"
- $f(x)$ — "A function f that takes an argument x"
- $f : \mathbb{R} \to \mathbb{R}$ — "A function f that takes reals and outputs reals"

Sum and Product Notation

- $\sum_{i=1}^{\infty} \frac{1}{i}$ — $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ...$
  "The sum from $i = 1$ to infinity of $\frac{1}{i}$"
- $\prod_{n=2}^{5} \frac{n}{n-1}$ — $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4}$
  "The product from $n = 2$ to 5 of $\frac{n}{n-1}$"

Imaginary and Complex Units

- i — "The imaginary unit defined as $\sqrt{-1}$"
- $a + bi$ — "A complex number, where $a, b \in \mathbb{R}$

Algebra Day 1 Notes

Math Circle Competition Team

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Polynomials

Definition: Let A be a set of elements with the property that for $a, b, c \in A$ we also have $a+b, a-b, a \cdot b \in A$ and $a(b+c) = ab + ac$. Such a structure is called a **ring**. We will only consider the rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$.

A **polynomial** over A is an expression of the form

$$P(X) = a_0 + a_1 X + ... + a_n X_n,$$

where X is a symbol (or **indeterminate**) and $a_0, a_1, ...a_n \in A$ are called the **coefficients** of the polynomial P. The collection of all such polynomials is denoted by $A[X]$.

Note:

- The definition of a polynomial is different from that of a function. The rigorous definition of a polynomial is *slightly* beyond the scope of this course.
- Also, the set of values of $P(X)$ with $X \in A$ does not uniquely determine the coefficients a_j of P (as an example, take the polynomials 0 and $X^2 + X$ over $\mathbb{Z}_2$). This is not what we want, since we want to say that two polynomials are equal if and only if their coefficients are all equal. Therefore we cannot define a polynomial in terms of its values.

Addition: Let $P(X) = a_0 + a_1 + ... + a_n X_n$ and $Q(X) = b_0 + b_1 X + \ + b_m X_m$. Then $P(X) + Q(X) = (a_0 + b_0)X + (a_1 + b_1)X_1 + ...$ (pointwise addition of the coefficients).

Multiplication: Let $P(X) = a_0 + a_1 + ... + a_n X_n$ and $Q(X) = b_0 + b_1 X + \ + b_m X_m$. Then $P(X)Q(X) = c_0 + c_1 X^1 + ... + c_{m+n} X^{m+n}$ where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$ (consider $a_i = 0$ if $i > n$ and $b_j = 0$ if $j > m$). In school you may have learned this as the **FOIL** method.

The **degree** of a polynomial with coefficients $a_n, a_{n-1}, ..., a_0$ is the largest number k for which $a_k \neq 0$. (The degree of the zero polynomial is either left undefined, or chosen to be negative, usually $-\infty$).

In our case, if we impose $a_n \neq 0$, the degree of P is n and a_n is called the **leading coefficient** of P. A polynomial is called **monic** if its leading coefficient is 1.

To each polynomial we associate its corresponding **polynomial function** in the following way:

To $P(X) = a_0 + a_1X + ... + a_nX_n$, we associate the function $P : A \longmapsto A$ which sends an element $y \in A$ to $P(y) = a_0 + a_1y + ... + a_ny^n$.

Example: of $P(X) = X^2 + 3$ then then its corresponding polynomial function is $P(x) = x^2 + 3$ and we have for example $P(4) = 17$.

A **root** of a polynomial is a number x such that $a_0 + a_1x + ... + a_nx^n = 0$ i.e. $P(x) = 0$.

Fundamental Theorem of Algebra: Every non-constant single-variable polynomial with complex coefficients has at least one complex root. In particular, this implies that a non-zero polynomial of degree n with complex coefficients has exactly n roots (not necessarily distinct).
The proof involves some knowledge of Complex Analysis, so we omit it.

Corollary: If P and Q are two polynomials of degree n with complex roots which have the same roots and same leading coefficient, then they are equal.

- Try to prove this on your own! Use the fact that there are n roots of both polynomials, and those roots will also be the roots of $P(X) - Q(X)$, which will have degree at most $n - 1$ because the leading terms are the same and will cancel. Then finish with the Fundamental Theorem of Algebra.

As an example, we can see that every quadratic has exactly two roots x_1 and x_2 which satisfy $ax^2 + bx + c = a(x - x_1)(x - x_2)$.

The Division Algorithm

1. (For integers) If $a, b \in \mathbb{Z}$ and $b > 0$ then there exists $q, r \in \mathbb{Z}$ such that $a = bq + r$ and $0 \leq r < b$.
2. (For polynomials) If $f, g \in \mathbb{R}[x]$ and $g \neq 0$ then there exists $q, r \in \mathbf{r}[x]$ such that $f = q \cdot g + r$ and $r = 0$ or $\deg(r) < \deg(g)$.

We say that a polynomial $Q(X)$ divides a polynomial $P(X)$ if we can write

$$P(X) = Q(X) \cdot S(X) \text{ where } S(X) \text{ is a polynomial.}$$

In particular, this shows that when x is a root of Q, it is also a root of P.

Examples: $Q(X) = X + 4$ divides $P(X) = X^2 + 3X - 4$, as $P(X) = (X + 4)(X - 1) = Q(X)(X - 1)$. However, Q does not divide $T(X) = X^2 - 3X + 2$ since $T(X) = (X - 1)(X - 2)$ and -4 is a root of Q which is not a root of T.

Remark: When we study division, we must be careful over which set the polynomials are defined. For example, $2X$ divides X^2 in $\mathbb{Q}[X]$, but not in $\mathbb{Z}[X]$.

Factor Theorem: The number r is a root of the polynomial $P(X)$ if and only if $(X - r) \mid P(X)$.

- Try to prove this on your own! This is an **if and only if** statement, also known as a **bidirectional** statement, or an **iff** statement, which means we to need to prove it in both the forward and reverse directions (left of the iff implies right and then right of the iff implies left). Use the Division Algorithm for one direction and the Fundamental Theorem of Algebra for another.

r is a **root of multiplicity** m of P if $(X - r)^m \mid P(X)$ but $(X - r)^{m+1} \nmid P(X)$.

Example: 1 is a root of multiplicity 2 of $P(X) = X^2 - 2X + 1 = (X - 1)^2$.

Rational Root Theorem: Let $P(X) = a_n X_n + a_{n-1} X_{n-1} + ... + a_1 X + a_0$, where P has integer coefficients. If a_0 and a_n are nonzero, then each rational root $x = \frac{p}{q}$, where p and q are relatively prime, satisfies $p \mid a_0$ and $q \mid a_n$.

- Try to prove this on your own! Consider a rational root $r = \frac{p}{q}$. we know that $P(\frac{p}{q}) = 0$, which we can then expand into the general form of a polynomial with exponents and coefficients. We can then make some arguments knowing p is relatively prime to q.

Complex Conjugate Root Theorem: If P is a polynomial with real coefficients, and $a + bi$ is a root of P, then its complex conjugate $a - bi$ is also a root of P.

- Try to prove this on your own! Factor P as $P(X) = a_n(X - x_1)(X - x_2)...(X - x_n)$. Then note that as P has real coefficients, its complex conjugate $\overline{P}$ should have the same coefficients.Then factor $\overline{P}$ in the same way and compare the expressions.

Corollary: If the degree of a polynomial with real coefficients is odd, then it must have at least one real root.

Worked Examples:

1. Find a polynomial f with integer coefficients such that $\frac{1}{2}$ and $-\frac{1}{5}$ are roots of f.

   Solution:
   $f(x) = (2x - 1)(5x + 1) = 10x^2 - 3x - 1$

2. Find a polynomial f with integer coefficients such that $\sqrt{2} + \sqrt{3}$ is a root of f.

   Solution:
   Say $x = \sqrt{2} + \sqrt{3}$. $x^2 = 5 + 2\sqrt{6}$. $(x^2 - 5)^2 = 24$. $f(x) = x^4 - 10x^2 + 1$.

3. Suppose f is a polynomial with integer coefficients such that $f(0)$ and $f(1)$ are odd numbers. Prove that f has no integer roots.

   Solution:
   Assume $P(a) = 0$. Then $P(x) = (x - a)Q(x)$. So $P(0) = -aQ(0)$ and $P(1) = (1 - a)Q(1)$. But one of the numbers a or $1 - a$ is even. So at least one of $P(0)$ or $P(1)$ must be even as well.

Quadratics

We now examine a very specific family of polynomials, namely, the quadratic polynomials. In contest and in practice, we are almost always interested in when a quadratic is equal to 0. Recall that a general quadratic equation has the form

$$ax^2 + bx + c = 0,$$

where a, b, c are usually real numbers and $a \neq 0$. To solve such equations, we begin by dividing by a on both sides to obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

$$x^2+\frac{b}{a}x+\frac{c}{a} = x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2 = \left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2 = \left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2} = 0.$$

The term $\Delta = b^2 - 4ac$ is known as the **discriminant** of the quadratic $ax^2 + bx + c$. So now we have

$$ax^2 + bx + c \Leftrightarrow \left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2} = 0.$$

This shows that the equation has two distinct solutions (but we now know these as roots) when $\Delta \geq 0$, one **double root** (which we know as a root with multiplicity 2) when $\Delta = 0$, and no real roots when $\Delta < 0$. Recall the factorizations of the difference and sum of squares

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^2 + b^2 = (a + bi)(a - bi)$$

Which we can use when $\Delta \geq 0$ to see that the roots are given by

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a},\ x_2 = \frac{-b - \sqrt{\Delta}}{2a}.$$

and when $\Delta < 0$ to see that the roots are given by

$$x_1 = \frac{-b + i\sqrt{-\Delta}}{2a},\ x_2 = \frac{-b - i\sqrt{-\Delta}}{2a}.$$

You might already know this formula well as the **quadratic formula**! There are also formulas for the roots of cubic and quartic polynomials, but they are much more complicated, and are rarely practical. An important result from abstract algebra (the Abel-Ruffini theorem) states that there is no general formula expressible in terms of elementary functions for the roots of polynomials of degree 5 and higher. Therefore, it is often pragmatic to try and reduce higher degree equations to (possibly multiple) quadratic ones via substitution. (Even Cardanos formula for cubic polynomials employs this technique, reducing the cubic to a quadratic).

Sign of a Quadratic: Let $f : \mathbb{R} \to \mathbb{R}. f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$ and $a \neq 0$. We are interested in how the sign of $f(x)$ (that is, positive or negative sign) changes are x varies over the real numbers.
Recall from above that we have $ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$.
This shows that for all $x \in \mathbb{R}$, $\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2} \geq -\frac{\Delta}{4a^2}$ with equality when $x = -\frac{b}{2a}$.

If $a > 0$, we have and that $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] \geq -\frac{\Delta}{4a}$ and that $-\frac{\Delta}{4a}$ is the **minimum possible value** of $ax^2 + bx + c$.

If $a < 0$, we have and that $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] \leq -\frac{\Delta}{4a}$ and that $\frac{\Delta}{4a}$ is the **maximum possible value** of $ax^2 + bx + c$.

Let x_1 and x_2 be the roots of $ax^2 + bx + c = 0$. Recall that when $\Delta \geq 0$, the roots are real, so **without loss of generality** (**WLOG**) $x_1 < x_2$ (which we need for characterizing the overall sign of the quadratic). Notice that we can swap the inequality and our logic still holds. This is a powerful technique in proof-writing.

Recall by FTA that every quadratic factors as $ax^2 + bx + c = a(x - x_1)(x - x_2)$. The generalized sign of a quadratic is given as presented in the following table.

Δ	a	$ax^2 + bx + c$
< 0	> 0	$> 0 \, \forall x \in \mathbb{R}$
< 0	< 0	$< 0 \, \forall x \in \mathbb{R}$
≥ 0	> 0	< 0 if $x \in (x_1, x_2)$ and > 0 if $x \notin (x_1, x_2)$
≥ 0	< 0	> 0 if $x \in (x_1, x_2)$ and < 0 if $x \notin (x_1, x_2)$

Table 1: Sign of a quadratic

Graphing the Quadratic: The graph of a quadratic function is called a **parabola**. The important points on the graph are the following:

1. The intersection(s) with the real axis, i.e. the values of $f(x) = 0$, which are given by the roots x_1, x_2. So if the roots are not real, the parabola never crosses the x-axis.

2. The intersection with the y-axis, i.e. the value of $f(0)$, which for $f(x) = ax^2 + bx + c$ is just some constant c.

3. The vertex of the parabola, which has coordinates $-\frac{b}{2a}, -\frac{\Delta}{4a}$. This can be either the minimum or the maximum of the function, as described above.

The graph of a quadratic can be a very useful tool when solving problems, especially when knowing the relationship between the graph and one of the possible situations for the sign of a quadratic. As a short exercise, try to graph matching parabolas for each of the four situations detailed in the table above.

Worked Examples:

1. $(x+1)(x+2)(x+3)(x+4) = 5$. Solve for x.

   Solution:
   $(x+1)(x+2)(x+3)(x+4) = (x^2+5x+4)(x^2+5x+6)$. Substitute $y = x^2+5x$ and solve.

2. $\frac{x^2}{a} + \frac{ab^2}{x^2} = 2\sqrt{2ab}(\frac{x}{a} - \frac{b}{x})$ where $a, b \in \mathbb{R}_+$. Solve for x.

   Solution:
   Write as $a(\frac{x^2}{a^2} + \frac{b^2}{x^2}) = 2\sqrt{2ab}(\frac{x}{a} - \frac{b}{x})$, set $y = \frac{x}{a} - \frac{b}{x}$.

3. $\frac{7x}{2x^2-5x+4} + \frac{8x}{2x^2+3x+4} = 0$, solve for x.

   Solution: The denominators are always positive so only solution can be $x = 0$

4. $\frac{7x}{2x^2-5x-4} + \frac{8x}{2x^2+3x-4} = 0$, solve for x.

   Solution:
   $\frac{7}{2x-5-\frac{4}{x}} + \frac{8}{2x+3-\frac{4}{x}} = 0$
   Set $y = 2x - \frac{4}{x}$
   Solve $\frac{7}{y-5} + \frac{8}{y+3}$

5. Solve the system

$$\begin{cases} x_1^2 + ax_1 + (\frac{a-1}{2})^2 = x_2 \\ x_2^2 + ax_2 + (\frac{a-1}{2})^2 = x_3 \\ \vdots \\ x_n^2 + ax_n + (\frac{a-1}{2})^2 = x_1 \end{cases}$$

   Solution:
   Add the equations
   $(x_1 + \frac{a-1}{2})^2 + (x_2 + \frac{a-1}{2})^2 + ... + (x_n + \frac{a-1}{2})^2 = 0$

$$\begin{cases} ax^2 + (b+d)x + c = 0 \\ bx^2 + (c+d)x + a = 0 \\ cx^2 + (a+d)x + b = 0 \end{cases}$$

Prove that this system has a common solution x iff $a+b+c+d=0$ and $a<b<c<d$.

6. Solve for x:

$$x^3 - 2ax^2 + (a^2+1)x + 2 - 2a = 0, a > 2 \text{ is a real number.}$$

Solution:
Solve for a in terms of x.

$$a^2x + a(-2x-2) + x^3 + x + 2 = 0$$

$$\Delta = 4x^2 - 8x + 4$$

We get $a = \frac{x^2+x}{x}$ or $a = \frac{x^2-x-2}{x}$.

7. Can you have two quadratic functions ax^2+bx+c and $(a+1)x^2+(b+1)x+(c+1)$ where $a,b,c \in \mathbb{Z}$ such that both functions have integer roots?

Solution:
No. A quadratic can be written as $a(x-x_1)(x-x_2) = a(x^2 - x_1x - x_2x + x_1x_2)$. Use a parity argument to show it not not possible for both ax^2+bx+c and $(a+1)x^2+(b+1)x+(c+1)$ to have integer roots.

Vieta's Relations and Second-Order Homogeneous Linear Recurrences

Vieta's Relations on Quadratic Equations: Let $a,b,c \in \mathbb{R}$, $a \neq 0$, and x_1, x_2 be the roots of $ax^2+bx+c=0$. We have

$$x_1 + x_2 = -\tfrac{b}{a} \text{ and } x_1x_2 = \tfrac{c}{a}$$

Proof: (This time a proof will be provided) Recall that any quadratic can be factored such that

$$ax^2+bx+c = a(x-x_1)(x-x_2) = ax^2 - a(x_1+x_2)x + ax_1x_2.$$

This must hold for all x, so it follows that the coefficients for x, x^2, and the constant term are all the same. By comparing these coefficients, we obtain the relations as stated.

Vieta's Relations on Cubic Equations: Let $a,b,c,d \in \mathbb{R}$, $a \neq 0$, and x_1, x_2, x_3 be the roots of $ax^3+bx^2+cx+d=0$. We have

$$x_1+x_2+x_3 = -\tfrac{b}{a},\ x_1x_2+x_1x_3+x_2x_3 = \tfrac{c}{a} \text{ and } x_1x_2x_3 = -\tfrac{d}{a}$$

We will soon discover that there is a generalized form of Vieta's relations for all such polynomials of arbitrary degree.

Second-Order Homogeneous Linear Recurrences: A second order linear recurrence equation is a recurrence equation on a sequence of numbers $\{x_n\}_n \geq 0$ which expresses x_n as a first-degree polynomial in terms of x_{n-1} and x_{n-2}:

$$x_n = Ax_{n-1} + Bx_{n-2},$$

for some given constants A and B. To solve for x_n, we require initial conditions $x_0 = U$ and $x_1 = V$, and we determine the generalized form of x as follows:

We assign the recurrence relation $x_n = Ax_{n-1} + Bx_{n-2}$ a **characteristic polynomial** $r^n = Ar^{n-1} + Br^{n-2}$. Note the relationship between the exponents of the characteristic polynomial and the subscripts of the recurrence relation. Now divide both sides of the equation by r^{n-2} and obtain

$$r^2 = Ar + B.$$

We wish to solve for the **characteristic roots** of this polynomial r_1 and r_2. We classify the solutions as follows:

If $r_1 \neq r_2$ then $x_n = C_1 r_1^n + C_2 r_2^n$ for some constants C_1 and C_2.
If $r_1 = r_2 = r$ then $x_n = C_1 r^n + C_2 n r^n$.
In either case, we determine the constants by looking at the initial conditions. For example, in the case that $r_1 \neq r_2$ we have

$$\begin{cases} C_1 + C_2 = U \\ C_1 r_1 + C_2 r_2 = V \end{cases}$$

and we proceed similarly for the $r_1 = r_2$ case.
Note that the method of raising the subscripts to the exponents of a characteristic polynomial can be done to theoretically solve any order homogeneous linear recurrence relation. However, difficulties naturally arise with solving higher degree characteristic polynomials.

Worked Examples:

1. Given the recurrence relation on the left, verify that the Fibonacci Numbers can be defined as the generalized form on the right:

$$F_n = F_{n-2} + F_{n-1},\ (1, 1, 2, 3, 5, 8, ...),\ F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

   Solution: Straightforward

2. Find the general term of the sequence given by $x_0 = 6, x_1 = 20$, and

$$(n+1)(n+2)x_n = 5(n+1)(n+3)x_{n-1} - 6(n+2)(n+3)x_{n-2}$$

Solution:

$$\frac{x_n}{n+3} = 5 \cdot \frac{x_{n-1}}{n+2} - 6 \cdot \frac{x_{n-2}}{n+1}$$

$$\text{Set } y_n = \frac{x_n}{n+3}, y_{n-1} = \frac{x_{n-1}}{n+2}, y_{n-2} = \frac{x_{n-2}}{n+1}.$$

$$y_n = 5y_{n-1} - 6y_{n-1} \Rightarrow x^2 - 5x + 6 = 0$$

$\lambda_{1,2} = 2, 3$. $y_n = 3^n k_1 + 2^n k_2$, $y_0 = 2$, $y_1 = 5 \Rightarrow k_1 = 1, k_2 = 1$.

3. How many ways can we tile a $2 \times n$ rectangle using 2×1 domino pieces?

   Solution:
   Let a_n denote the number of ways we can tile $2 \times n$ with $2 \times n$ dominoes. $a_n = a_{n-1} + a_{n-2}$, $a_n = F_n$.

4. How many ways can we tile a $3 \times 2n$ rectangle using 2×1 domino pieces?

   Solution:

   $$U_n = U_{n-1} + U_{n-1} + U_{n-1} + V_n + V_n,$$

   where U_n is the total number of ways to tile a $2 \times n$ rectangle and V_n is the total number of ways to tile a $(2n-3) \times 3$ rectangle with a corner tile removed.

   $$U_n = 3U_{n-1} + 2V_n$$
   $$V_n = U_{n-2} + V_{n-1}$$

Generalized Vieta's, Symmetric Polynomials, and Newtons Sums

Definitions: A polynomial in n variables is a polynomial which has n indeterminates $X_1, ...X_n$ (we can also label them as $X, Y, Z...$).

For example, $f(X,Y) = X^3 + 2XY - Y^2 + 5$ is a polynomial in two variables.

A polynomial P in n variables is called **symmetric** if whenever σ is a permutation (a rearrangement where order matters) of elements of the set $1, 2..., n$ we have $P(X_1, X_2, ...X_n) = P(X_{\sigma(1)}, X_{\sigma(2)}, ...X_{\sigma(n)})$.

Example: $g(X,Y) = 3X^2 - 2XY + 3Y^2$ is symmetric as $g(X,Y) = g(Y,X)$, but $h(X,Y,Z) = X^3 + 3XY - 2XZ + Z^2 - Y^2$ is not, since $h(Y,Z,X) \neq h(X,Y,Z)$.

For any $1 \leq r \leq n$ let

$$S_r(X_1, X_2, ..., X_n) = \sum_{1 \leq i_1 < i_2 < ... < i_r \leq i_n} X_{i_1} X_{i_2} ... X_{i_r}$$

be the r^{th} **elementary symmetric polynomial** in $X_1, X_2, ..., X_n$.

Example: $S_1(X_1, X_2, X_3) = X_1 + X_2 + X_3$
$S_2(X_1, X_2, X_3) = X_1X_2 + X_2X_3 + X_1X_3$
$S_3(X_1, X_2, X_3) = X_1X_2X_3$.

For $1 \leq r \leq n$ define the n^{th} **power sum** as $P_r(X_1, X_2, ..., X_n) = \sum_{k=1}^{n} X_k^r$.

Generalized Vieta's Relations: If $x_1, x_2, ..., x_n$ are the roots of the polynomial $a_nX^n + a_{n-1}X^{n-1} + ... + a_1X + a_0$ then for each $1 \leq r \leq n$:

$$S_r(x_1, x_2, ..., x_n) = (-1)^r \frac{a_{n-r}}{a_n}.$$

Proof: (This proof will also be provided) Recall that

$$a_nX^n + a_{n-1}X^{n-1} + ... + a_1X + a_0 = a_n(X - x_1)(X - x_2)...(X - x_n).$$

Expand the right hand side and compare coefficients to the left hand side. You obtain exactly Vieta's relations.

Fundamental Theorem of Symmetric Polynomials: Any symmetric polynomial in $X_1, X_2, ..., X_n$ can be expressed as a polynomial in the elementary symmetric polynomials in $X_1, X_2, ..., X_n$.

Example: $\sum_{1 \leq i \neq j \leq n} X_i^3 X_j = S_1^2 S_2 - 2S_2^2 - S_1S_3 + 4S_4$ where $S_k = S_k(X_1, X_2, ..., X_n)$.

Newton's Identities: Newton Sums give us an easy way to calculate the sum of the powers of the roots of any given polynomial. For each $1 \leq r \leq n$

$$P_r - S_1P_{r-1} + S_2P_{r-2} - ... + (-1)^{n-1}S_{n-1}P_{r-n+1} + (-1)^n S_n P_{r-n} = 0$$

where P_k and S_k stand for the k^{th} power sum and the k^{th} elementary symmetric polynomial of $X_1, X_2, ..., X_n$, respectively.

Remark: The second sum is a special case of the first sum, for $j > k$ and when S_j in k variables is defaulted to 0.

Proof: All of Newton's identities for $r \geq n$ can be proven easily with Vieta's relations, so this is left as an exercise for the reader.

Worked Examples:

1. Solve the system of equations

$$\begin{aligned} x+y+z &= 6 \\ x^2+y^2+z^2 &= 14 \\ x^3+y^3+z^3 &= 36 \end{aligned}$$

   Solution: From the given equations we find that $xy+yz+zx = 11$ and $xyz = 6$, using the identities $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$ and $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$. Then by Vietas, x, y, z are the roots of $P(X) = X^3-6X^2+11X-6 = (X-1)(X-2)(X-3)$, hence $x, y, z = 1, 2, 3$.

2. Solve the system of equations

$$\begin{aligned} x+y+z &= 4 \\ x^2+y^2+z^2 &= 14 \\ x^3+y^3+z^3 &= 34 \end{aligned}$$

   Solution: Let x, y, z be the roots of the equation $t^3-at^2+bt-c = 0$. From the first equation we obtain $a = 4$, from the first two $b = 1$. Also $P_3-aP_2+bP_1-cP_0 = 0$ yields $34-4\cdot 14+1\cdot 4-3c = 0$, hence $c = -6$. We deduce that x, y, z are the roots of the equation $t^3-4t^2+t+6 = 0$. The roots are $-1, 2, 3$.

3. **USAMO 1984 #1:** In the polynomial $x^4-18x^3+kx^2+200x-1984 = 0$, the product of two of its roots is -32. Find k.

   Solution: Using Vieta's formulas, we have:

$$\begin{aligned} a+b+c+d &= 18, \\ ab+ac+ad+bc+bd+cd &= k, \\ abc+abd+acd+bcd &= -200, \\ abcd &= -1984. \end{aligned}$$

   From the last of these equations, we see that $cd = \frac{abcd}{ab} = \frac{-1984}{-32} = 62$. Thus, the second equation becomes $-32+ac+ad+bc+bd+62 = k$, and so $ac+ad+bc+bd = k-30$. The key insight is now to factor the left-hand side as a product of two binomials: $(a+b)(c+d) = k-30$, so that we now only need to determine $a+b$ and $c+d$ rather than all four of a, b, c, d. Let $p = a+b$ and $q = c+d$. Plugging our known values for ab and cd into the third Vieta equation, $-200 = abc+abd+acd+bcd = ab(c+d)+cd(a+b)$, we have $-200 = -32(c+d)+62(a+b) = 62p-32q$. Moreover, the first Vieta equation, $a+b+c+d = 18$, gives $p+q = 18$. Thus we have two linear equations in p and q, which we solve to obtain $p = 4$ and $q = 14$.

   Therefore, we have $(\underbrace{a+b}_{4})(\underbrace{c+d}_{14}) = k-30$, yielding $k = 4\cdot 14+30 = 86$.

Inequalities

AM-GM: For any positive integer n and **non-negative** real numbers $a_1, a_2, ..., a_n$ we have

$$\frac{a_1 + ... + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$$

with equality occurring iff $a_1 = a_2 = ... = a_n$. This is known as the **arithmetic mean-geometric mean inequality**. The arithmetic mean is also known by the names of "average" or sometimes simply "mean."

Proof: Most of the basic inequalities that will be introduced have standard proofs utilizing a concept called **mathematical induction**. Induction is an extremely powerful and widely used proof technique, but with the limited time in this course, there is sadly no room for a dedicated lesson on it. Any interested reader should certainly aim to add induction to their repertoire of mathematical tools at the earliest possible convenience. As one might have guessed, AM-GM is proven with induction, and as it is a foundational inequality, many other inequalities will be later proven with AM-GM.

Corollary (GM- HM): For any **positive** integers $a_1, ...a_n$ we have

$$\sqrt[n]{a_1 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + ... + \frac{1}{a_n}}$$

. The right hand side is known as the **harmonic mean**.

Proof: Immediate from AM-GM.

Corollary (Hölder's): For $a_{11}, ..., a_{1n}, a_{21}, ..., a_{2n}, ..., a_{k1}, ..., a_{kn}$**non-negative** reals we have

$$(a_{11} + ... + a_{1n})(a_{21} + ... + a_{2n})...(a_{k1} + ... + a_{kn}) \geq \sqrt[k]{a_{11}a_{21}...a_{k1}} + ... \sqrt[k]{a_{1n}a_{2n}...a_{kn}}$$

This is known as Hölder's inequality for sums.

Proof: Try to prove this on your own! Write out $S_1 = a_{11} + a_{12} + ... + a_{1n}, ..., S_k = a_{k1} + ...akn$. Then take the k^{th} root of the inequality and try to transform it into AM-GM.

Cauchy-Schwarz: Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ be real numbers. Then

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1b_1 + a_2b_2 + ...a_nb_n)^2$$

Equality occurs when $\frac{a_i}{b_i} = t$ for all i and some constant t or when $a_i = 0$ for all i or when $b_i = 0$ for all i.

Proof: For every real number t and every index i,

$$a_i^2 t^2 - 2a_i t b_i + b_i^2 = (a_i t - b_i)^2 \geq 0.$$

Summing these, we find that for all real numbers t

$$(a_1^2 + a_2^2 + ... + a_n^2)t^2 - (2a_1b_1 + a_2b_2 + ... + a_nb_n)t + (b_1^2 + b_2^2 + ... + b_n^2) \geq 0.$$

The discriminant of this equation is less than or equal to 0. The condition that the discriminant is non-positive translates into the Cauchy-Schwarz inequality.

Corollary (QM-AM): Let $a_1, a_2, ..., a_n$ be **non-negative** real numbers. Then

$$\sqrt{\frac{a_1^2 + a_2^2 + ...a_n^2}{n}} \geq \frac{a_1 + ... + a_n}{n}$$

The left hand side is known as the **quadratic mean**. It also also known as the "root-mean-square."

Proof: Direct application of C-S with the special case that $b_1 = b_2 = ... = b_n = 1$.

Corollary (Titu's Lemma): Let $a_1, a_2, ..., a_n$ be real numbers and let $b_1, b_2, ..., b_n > 0$. Then

$$\frac{a_1^2}{b_1^2} + \frac{a_2^2}{b_2^2} + ... + \frac{a_n^2}{b_n^2} \geq \frac{(a_1 + a_2 + ...a_n)^2}{b_1 + b_2 + ...b_n}.$$

Proof: Multiply both sides by $(b_1 + ... + b_n)$ Change the RHS to be under a radical and then apply C-S.

Worked Examples:

1. Prove that for every integer $n \geq 2$:

$$n! < \left(\frac{n+1}{2}\right)^n$$

   Solution: $\sqrt[n]{n!} < \frac{n+1}{2} = AM$

2. $\sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n} < 1 + \frac{1}{n+1}$

   Solution: take $a_1 = 1, a_2 = a_3 = ... = a_{n+1} = 1 + \frac{1}{n}$

3. Prove that
$$\left(\frac{a+b+c}{3}\right)^2 \leq \frac{a^2+b^2+c^2}{3}$$

Solution: Use Titu's Lemma with $x = y = z = 4$ or write as $(a+b+c)^2 \leq (a^2+b^2+c^2)(1+1+1)$, which is clear by Cauchy-Schwarz.

4. Let p be a polynomial with positive coefficients. Prove that $P(x^2)P(y^2) \geq P(xy)^2$

Solution: Let
$$P(x) = a_n x^n + .. + a_0$$
Then,
$$(a_n x^{2n} + a_{n-1}x^{2n-2} + .. + a_0)(a_n y^{2n} + a_{n-1}y^{2n-2} + .. + a_0) \geq (a_n(xy)^n + a_{n-1}(xy)^{n-1} + .. + a_0)^2$$
by Cauchy Schwarz.

5. $a_1 + ... + a_n = n$. Prove that $a_1^4 + ... a_n^4$.

Solution: Let $a_i^2 = b_i$. Then
$$(1^2 b_1^2 + ... + 1^2 b_n^2) \leq (b_1^2 + ... + b_n^2)(1^2 + ... + 1^2)$$
$$(a_1^2 + ... + a_n^2)^2 \leq (a_1^4 + ... + a_n^4)n$$
$$(a_1^2 + .. + a_n^2)(1^2 + ... + 1^2) \geq (a_1^2 + ... + a_n^2)^2 = n$$
$$n(a_1^2 + ... + a_n^2) \geq n^2$$
$$(a_1^2 + ... + a_n^2) \geq n$$

6. Let $a_1, ..., a_n$ be distinct real numbers. Let $b_1, ..., b_n$ be a permutation of a_i. Find the max of $a_1 b_1 + ... + a_n b_n$ over all the possible permutations.

Solution: $(a_1 b_1 + ... + a_n b_n)^2 \leq (a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2) = (a_1^2 + .. + a_n^2)^2$
$\Rightarrow (a_1 b_1 + ... + a_n b_n) \leq a_1^2 + .. + a_n^2$, equality occurs when $a_i = b_i$.

7. Prove the equality case for Cauchy-Schwarz

Solution:

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2$$
$$a_1^2 t^2 - 2a_1 b_1 t + b_1^2 > 0$$
$$t^2(a_1^2 + \cdots + a_n^2) - 2t(a_1 b_1 + \cdots + a_n b_n) + (b_1^2 + \cdots + b_n^2) \geq 0$$

$\Delta = 0$ when $a_1^2 t^2 - 2a_1 b_1 t + b_1^2 = \cdots = a_n^2 t^2 - 2a_n b_n t + b_n^2 = 0 \rightarrow b_1 = a_1 t,\ b_2 =$

$a_2t, ..., b_n = a_nt$ for some t.

8. Consider the real numbers $x_1 \geq x_2 \geq ... \geq x_n$. Prove that

$$x_0 + \frac{1}{x_0 - x_1} + \cdots + \frac{1}{x_{n-1} - x_n} \geq x_n + 2n$$

Solution: Using Titu's Lemma:

$$x_0 + \frac{1}{x_0 - x_1} + \cdots + \frac{1}{x_{n-1} - x_n} \geq x_n + 2n \geq x_0 + \frac{n^2}{x_0 - x_n} \geq x_n + 2n$$

$$x_0 - x_n + \frac{n^2}{x_0 - x_n} - 2n \geq 0$$

$$(x\sqrt{x_0 - x_n} - \frac{n}{\sqrt{x_0 - x_n}})^2 \geq 0$$

Equality occurs when $x_{n-i} = x_{n+i}$

Sequences and Series

Arithmetic Sequence: An arithmetic sequence has a common difference between terms. For instance, $\{1, 4, 7, 10, ...\}$ is an arithmetic sequence. If a_n is an arithmetic sequence with first term a and common difference d, then

$$a_1 + a_2 + a_3 + ... + a_n = \frac{n}{2}(a_1 + a_n).$$

For a sequence with n terms, n odd, the middle term of the sequence is $\frac{(a_1 + a_n)}{2}$.

Geometric Sequence: A geometric sequence has a common ratio between terms. For instance, $\{1, 2, 4, 8, 16, ...\}$ is a geometric sequence. If $|r| < 1$, then we have

$$1 + r + r^2 + r^3 + ... = \frac{1}{1 - r}.$$

Similarly, if we have a finite geometric sum with common ratio r, then

$$1 + r + r^2 + ... + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

Recursive Sequence: A recursively-defined sequence is one in which the nth term a_n is defined in terms of the previous terms $a_1, a_2, ..., a_{n-1}$. A common example is the *Fibonacci sequence*, in which $f_1 = 0$, $f_2 = 1$, and

$$f_n = f_{n-1} + f_{n-2}.$$

Thus the sequence is $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...\}$

Telescoping Series: A telescoping series is an infinite series whose terms cancel, leaving a finite number of terms. Below is a common example of a telescoping series.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1.$$

Useful Sum Formulas: (bonus exercise is to try and prove these)

- $1 + 2 + 3 + ... + n = \frac{(n)(n+1)}{2}$
- $1 + 3 + 5 + ... + (2n-1) = n^2$
- $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{(n)(n+1)(2n+1)}{6}$