

# Algebra Day 1 Problem Set

Math Circle Competition Team

September 3rd, 2017

## Polynomials

1. Prove that  $\sqrt{2}$  is irrational.
2. Find a polynomial with integer coefficients that has the zero  $1 + \sqrt{2}$ . Show that this number is irrational. Do the same for  $\sqrt{2} + \sqrt[3]{3}$ .
3. \* Given that  $x = 2$  is a root of  $P(X) = X^3 + 2X^2 - 5X - 6$ , find the other two roots.
4. \* Determine  $a, b$  so that  $(X - 1)^2$  divides  $aX^4 + bX^3 + 1$ .
5. **AMC 10A 2017 #24** For certain real numbers  $a, b$ , and  $c$ , the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of  $g(x)$  is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is  $f(1)$ ?

6. Is there a nonconstant polynomial  $f(x)$  such that

$$xf(y) + yf(x) = (x + y)f(x)f(y)$$

for all real numbers  $x$  and  $y$ ?

7. Find all the polynomials  $f \in \mathbb{Z}[X]$  that satisfy

$$2f(2X) = f(3X) + f(X).$$

8. \* **HMMT Feb 2015 Algebra #1** Let  $Q$  be a polynomial

$$Q(x) = a_0 + a_1x + \dots + a_nx^n$$

where  $a_0, a_1, \dots, a_n$  are non-negative integers. Given that  $Q(1) = 4$  and  $Q(5) = 152$ , find  $Q(6)$ .

9. **AIME I 2015 #10** Let  $f(x)$  be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find  $|f(0)|$ .

10. **AIME II 2016 #6** For polynomial  $P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2$ , define  $Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i$ . Then  $\sum_{i=0}^{50} |a_i| = \frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
11. **AIME I 2016 #11** Let  $P(x)$  be a nonzero polynomial such that  $(x - 1)P(x + 1) = (x + 2)P(x)$  for every real  $x$ , and  $(P(2))^2 = P(3)$ . Then  $P(\frac{7}{2}) = \frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
12. **HMMT Feb 2017 Algebra and NT #1** Let  $Q(x) = a_0 + a_1x + \dots + a_nx^n$  be a polynomial with integer coefficients, and  $0 \leq a_i < 3$  for all  $0 \leq i \leq n$ . Given that  $Q(\sqrt{3}) = 20 + 17\sqrt{3}$ , compute  $Q(2)$ .

## Quadratics

1. \* After making a suitable substitution, reduce the following equation to a quadratic and then solve it explicitly:

$$x(x + 1)(x + 2)(x + 3) = 24.$$

2. Solve the equation

$$x^3 + (x + 1)^3 + (x + 2)^3 + (x + 3)^3 = 0.$$

3. Solve the equation

$$x^4 - 97x^3 + 2012x^2 - 97x + 1 = 0.$$

4. **AIME I 2014 #6** The graphs  $y = 3(x - h)^2 + j$  and  $y = 2(x - h)^2 + k$  have y-intercepts of 2013 and 2014, respectively, and each graph has two positive integer x-intercepts. Find  $h$ .

5. **AIME I 2014 #14** Let  $m$  be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4$$

There are positive integers  $a$ ,  $b$ , and  $c$  such that  $m = a + \sqrt{b + \sqrt{c}}$ . Find  $a + b + c$ .

6. **HMMT Nov 2016 Team #10** Determine the largest integer  $n$  such that there exist monic quadratic polynomials  $p_1(x), p_2(x), p_3(x)$  with integer coefficients so that for all integers  $i \in [1, n]$  there exists some  $j \in [1, 3]$  and  $\min\mathbb{Z}$  such that  $p_j(m) = i$ .

## Vieta's and Second-Order Recurrences

1. Let  $a_n$  be a sequence of real numbers such that  $a_0 = 2$ ,  $a_1 = 21$ , and  $a_n = 6a_{n-1} - 9a_{n-2}$  for all  $n \geq 2$ . Find an explicit formula for  $a_n$ .

2. \* Find the sum of the squares of the roots of the equation  $x^2 - 5x + 4 = 0$ .
3. Determine  $(r + s)(s + t)(t + r)$  if  $r, s, t$  are the three roots of the polynomial  $x^3 + 9x^2 - 9x - 8$ .
4. **AIME II 2015 #6** Steve says to Jon, "I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form  $P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c$  for some positive integers  $a$  and  $c$ . Can you tell me the values of  $a$  and  $c$ ?"
- After some calculations, Jon says, "There is more than one such polynomial."
- Steve says, "You're right. Here is the value of  $a$ ." He writes down a positive integer and asks, "Can you tell me the value of  $c$ ?"
- Jon says, "There are still two possible values of  $c$ ."
- Find the sum of the two possible values of  $c$ .
5. **AIME I 2014 #9** Let  $x_1 < x_2 < x_3$  be the three real roots of the equation  $\sqrt{2014}x^3 - 4029x^2 + 2 = 0$ . Find  $x_2(x_1 + x_3)$ .
6. **AIME II 2014 #5** Real numbers  $r$  and  $s$  are roots of  $p(x) = x^3 + ax + b$ , and  $r + 4$  and  $s - 3$  are roots of  $q(x) = x^3 + ax + b + 240$ . Find the sum of all possible values of  $|b|$ .

## Newton Sums and Symmetric Polynomials

1. Find the zeros of the polynomial  $P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$  knowing that the sum of two of them is 4.
2. Solve the system of equations

$$\begin{aligned}x + y + z &= 5 \\ \frac{x}{zy} + \frac{y}{xz} + \frac{z}{xy} &= \frac{9}{4} \\ x^3 + y^3 + z^3 - 3xyz &= 5.\end{aligned}$$

3. Solve the system of equations Solve the system

$$\begin{aligned}x + y + z &= 5 \\ x(y + z)^2 + y(x + z)^2 + z(x + y)^2 &= -14 \\ x^2(y + z) + y^2(x + z) + z^2(x + y) &= 34.\end{aligned}$$

4. **HMMT Nov 2016 Team #3** Complex number  $\omega$  satisfies  $\omega^5 = 2$  Find the sum of all possible values of

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1.$$

5. **IMO 1988 #4** Show that the solution set of the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose length is 1988.

## Inequalities

1. \* Let  $a, b, c$  be positive numbers. Prove that

$$(a+b)(b+c)(c+a) \geq 8abc$$

2. Let  $x, y, z$  be positive numbers. Prove that

$$xyz \geq (x+y-z)(y+z-x)(z+x-y)$$

3. Prove that for any non-negative real numbers  $a, b, c$ :

$$a^3 + b^3 + c^3 + ab^2 + bc^2 + ca^2 \geq 2(a^2b + b^2c + c^2a)$$

4. Which one is larger:

$$\sqrt[4]{5} + \sqrt[4]{8} \text{ or } \sqrt[4]{6} + \sqrt[4]{7}?$$

5. Which one is larger:

$$17^{15} \text{ or } 31^{12}?$$

6. Find the triples of positive real numbers  $(x, y, z)$  such that

$$\frac{2x + 2y + z}{\sqrt{x^2 + y^2 + z^2}} = 3.$$

7. Prove that for any three positive real numbers  $a_1, a_2, a_3$ ,

$$\frac{a_1^2 + a_2^2 + a_3^2}{a_1^3 + a_2^3 + a_3^3} \geq \frac{a_1^3 + a_2^3 + a_3^3}{a_1^4 + a_2^4 + a_3^4}.$$

8. Let  $f_1, f_2, \dots, f_n$  be positive real numbers. Prove that for any real numbers  $x_1, x_2, \dots, x_n$ , the quantity

$$f_1x_1^2 + f_2x_2^2 + \dots + f_nx_n^2 - \frac{(f_1x_1 + f_2x_2 + \dots + f_nx_n)^2}{f_1 + f_2 + \dots + f_n}$$

is nonnegative.

9. **AIME II 2016 #15** For  $1 \leq i \leq 215$  let  $a_i = \frac{1}{2^i}$  and  $a_{216} = \frac{1}{2^{215}}$ . Let  $x_1, x_2, \dots, x_{216}$  be positive real numbers such that  $\sum_{i=1}^{216} x_i = 1$  and  $\sum_{1 \leq i < j \leq 216} x_i x_j = \frac{107}{215} + \sum_{i=1}^{216} \frac{a_i x_i^2}{2(1 - a_i)}$ . The maximum possible value of  $x_2 = \frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .

## Sequences and Series

- \* Evaluate the following series:
  - $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots = ?$
  - $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{99 \cdot 100} = ?$
  - $\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \dots = ?$
- In an arithmetic sequence of real numbers, the sum of the first two terms is 7 and the sum of the first six terms is 93. What is the sum of the first four terms?
- AMC 10A 2004 #24** Let  $a_1, a_2, \dots$  be a sequence with  $a_1 = 1$  and  $a_{2n} = n \cdot a_n$  for any positive integer  $n$  which is a power of 2. What is the value of  $a_{2^{100}}$ ?
- AIME I 2012 #2** The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the  $k$ th term is increased by the  $k$ th odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.
- AIME 1985 #1** Let  $x_1 = 97$ , and for  $n > 1$ , let  $x_n = \frac{n}{x_{n-1}}$ . Calculate the product  $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$ .
- AIME II 2005 #3** An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. Find the common ratio of the original series.
- AIME II 2008 #6** The sequence  $\{a_n\}$  is defined by  $a_0 = 1, a_1 = 1$ , and  $a_n = a_{n-1} + \frac{a_{n-1}^2}{a_{n-2}}$  for  $n \geq 2$ . The sequence  $\{b_n\}$  is defined by  $b_0 = 1, b_1 = 3$ , and  $b_n = b_{n-1} + \frac{b_{n-1}^2}{b_{n-2}}$  for  $n \geq 2$ . Find  $\frac{b_{32}}{a_{32}}$ .