Algebra Day 1 Solutions

Math Circle Competition Team

September 3rd, 2017

Polynomials

1. Prove that $\sqrt{2}$ is irrational.

   Solution:
   Note that $\sqrt{2}$ is a root of $x^2 - 2 = 0$, but it is not one of the candidate roots output by the rational root theorem. Therefore, it is irrational.

2. Find a polynomial with integer coefficients that has the zero $1 + \sqrt{2}$. Show that this number is irrational. Do the same for $\sqrt{2} + \sqrt[3]{3}$.

   Solution:
   $1 + \sqrt{2}$ fails the rational root test for $x^2 - 2x - 1 = 0$.
   Let $a = \sqrt{2} + \sqrt[3]{3}$. Then $a^2 = 2 + 2\sqrt{2}\sqrt[3]{3} + \sqrt[3]{9}$. We compute

   $$\sqrt[3]{9} = a^2 - 2 - 2\sqrt{2}\sqrt[3]{3} = a^2 - 2 - 2\sqrt{2}(a - \sqrt{2}) = a^2 + 2 - 2\sqrt{2}a.$$

*3. Given that $x = 2$ is a root of $P(X) = X^3 + 2X^2 - 5X - 6$, find the other two roots.

   Solution:

   $$\begin{array}{r|rrrr} & x^2 & +4x & +3 & \\ \hline x-2 & x^3 & +2x^2 & -5x & -6 \\ & -x^3 & +2x^2 & & \\ \hline & & 4x^2 & -5x & \\ & & -4x^2 & +8x & \\ \hline & & & 3x & -6 \\ & & & -3x & +6 \\ \hline & & & & 0 \end{array}$$

   Solving for $x^2 + 4x + 3 = 0$, we can find $\boxed{x = -3, -1}$.

*4. Determine a, b so that $(X - 1)^2$ divides $aX^4 + bX^3 + 1$.

   Solution:
   idea: do polynomial long division for $\frac{ax^4+bx^3+1}{x-1}$ and for $\frac{ax^4+bx^3+1}{(x-1)^2}$; you want your remainders for both to be equal to zero, so you can set up a system of equations (2

equations, 2 unknowns) and solve for a and b

5. **AMC 10A 2017 #24** For certain real numbers a, b, and c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

Solution:
$f(x)$ must have four roots, three of which are roots of $g(x)$. Using the fact that every polynomial has a unique factorization into its roots, and since the leading coefficient of $f(x)$ and $g(x)$ are the same, we know that

$$f(x) = g(x)(x - r)$$

where $r \in \mathbb{C}$ is the fourth root of $f(x)$. Substituting $g(x)$ and expanding, we find that

$$\begin{aligned} f(x) &= (x^3 + ax^2 + x + 10)(x - r) \\ &= x^4 + (a - r)x^3 + (1 - ar)x^2 + (10 - r)x - 10r. \end{aligned}$$

Comparing coefficients with $f(x)$, we see that

$$\begin{aligned} a - r &= 1 \\ 1 - ar &= b \\ 10 - r &= 100 \\ -10r &= c. \end{aligned}$$

We shall solve for only a and r. Since $10 - r = 100$, $r = -90$, and since $a - r = 1$, $a = -89$. Then,

$$\begin{aligned} f(1) &= (1 + r)(x^3 + ax^2 + x + 10) \\ &= (91)(-77) \\ &= -7007. \end{aligned}$$

6. Is there a nonconstant polynomial $f(x)$ such that

$$xf(y) + yf(x) = (x + y)f(x)f(y)$$

for all real numbers x and y?

7. Find all the polynomials $f \in \mathbb{Z}[X]$ that satisfy

$$2f(2X) = f(3X) + f(X).$$

*8. **HMMT Feb 2015 Algebra #1** Let Q be a polynomial

$$Q(x) = a_0 + a_1x + ...a_nx^n$$

where $a_0, a_1, ..., a_n$ are non-negative integers. Given that $Q(1) = 4$ and $Q(5) = 152$, find $Q(6)$.

Solution:
Since each a_i is a nonnegative integer, $152 = Q(5) \equiv a_0 (\text{mod } 5)$ and $Q(1) = 4 \implies a_i \leq 4$ for each i. Thus, $a_0 = 2$. Also, since $5^4 > 152 = Q(5), a_4, a_5, \ldots, a_n = 0$.
Now we simply need to solve the system of equations

$$5a_1 + 5^2a_2^2 + 5^3a_3^3 = 150$$
$$a_1 + a_2 + a_3 = 2$$

to get

$$a_2 + 6a_3 = 7.$$

Since a_2 and a_3 are nonnegative integers, $a_2 = 1$, $a_3 = 1$, and $a_1 = 0$. Therefore, $Q(6) = 6^3 + 6^2 + 2 = \boxed{254}$.

9. **AIME I 2015 #10** Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

10. **AIME II 2016 #6** For polynomial $P(x) = 1 - \dfrac{1}{3}x + \dfrac{1}{6}x^2$, define $Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) =$ $\sum_{i=0}^{50} a_ix^i$. Then $\sum_{i=0}^{50} |a_i| = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. **AIME I 2016 #11** Let $P(x)$ be a nonzero polynomial such that $(x - 1)P(x + 1) = (x + 2)P(x)$ for every real x, and $(P(2))^2 = P(3)$. Then $P(\frac{7}{2}) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

12. **HMMT Feb 2017 Algebra and NT #1** Let $Q(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integer coefficients, and $0 \leq a_i < 3$ for all $0 \leq i \leq n$.

Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$.

Quadratics

*1. After making a suitable substitution, reduce the following equation to a quadratic and then solve it explicitly:

$$x(x+1)(x+2)(x+3) = 24.$$

Solution:
basic idea: expand, substitution trick, solve quadratics

$$\begin{aligned} x(x+1)(x+2)(x+3) &= 24 \\ x^4 + 6x^3 + 11x^2 + 6x - 24 &= 0 \\ (x^2+3x)^2 + 2(x^2+3x) - 24 &= 0 \\ \big((x^2+3x)+6\big)\big((x^2+3x)-4\big) &= 0\big(x^2+3x+6\big)\big((x+4)(x-1)\big) &= 0 \end{aligned}$$

So then,

$$\big((x^2+3x)+6\big) = 0 \text{ or } \big((x+4)(x-1)\big) = 0$$

Therefore, $\boxed{x = -4, 1, \dfrac{-3 \pm i\sqrt{15}}{2}}$.

2. Solve the equation

$$x^3 + (x+1)^3 + (x+2)^3 + (x+3)^3 = 0.$$

3. Solve the equation

$$x^4 - 97x^3 + 2012x^2 - 97x + 1 = 0.$$

4. **AIME I 2014 #6** The graphs $y = 3(x-h)^2 + j$ and $y = 2(x-h)^2 + k$ have y-intercepts of 2013 and 2014, respectively, and each graph has two positive integer x-intercepts. Find h.

5. **AIME I 2014 #14** Let m be the largest real solution to the equation
$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4$
There are positive integers a, b, and c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a + b + c$.

6. **HMMT Nov 2016 Team #10** Determine the largest integer n such that there exist monic quadratic polynomials $p_1(x), p_2(x), p_3(x)$ with integer coefficients so that for all integers $i \in [1, n]$ there exists some $j \in [1, 3]$ and $min\mathbb{Z}$ such that $p_j(m) = i$.

Vieta's and Second-Order Recurrences

1. Let a_n be a sequence of real numbers such that $a_0 = 2, a_1 = 21$, and $a_n = 6a_{n-1} - 9a_{n-2}$ for all $n \geq 2$. Find an explicit formula for a_n.

*2. Find the sum of the squares of the roots of the equation $x^2 - 5x + 4 = 0$.

   Solution:
   ah just use quad formula and get your two roots and square them and sum them? is there a faster way??

3. Determine $(r+s)(s+t)(t+r)$ if r, s, t are the three roots of the polynomial $x^3 + 9x^2 - 9x - 8$.

4. **AIME II 2015 #6** Steve says to Jon, "I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form $P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c$ for some positive integers a and c. Can you tell me the values of a and c?"

   After some calculations, Jon says, "There is more than one such polynomial."

   Steve says, "You're right. Here is the value of a." He writes down a positive integer and asks, "Can you tell me the value of c?"

   Jon says, "There are still two possible values of c."

   Find the sum of the two possible values of c.

5. **AIME I 2014 #9** Let $x_1 < x_2 < x_3$ be the three real roots of the equation $\sqrt{2014}x^3 - 4029x^2 + 2 = 0$. Find $x_2(x_1 + x_3)$.

6. **AIME II 2014 #5** Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

Newton Sums and Symmetric Polynomials

1. Find the zeros of the polynomial $P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$ knowing that the sum of two of them is 4.

2. Solve the system of equations

$$x + y + z = 5$$
$$\frac{x}{zy} + \frac{y}{xz} + \frac{z}{xy} = \frac{9}{4}$$
$$x^3 + y^3 + z^3 - 3xyz = 5.$$

3. Solve the system of equations Solve the system

$$\begin{aligned} x+y+z&=5\\ x(y+z)^2+y(x+z)^2+z(x+y)^2&=-14\\ x^2(y+z)+y^2(x+z)+z^2(x+y)&=34. \end{aligned}$$

4. **HMMT Nov 2016 Team #3** Complex number ω satisfies $\omega^5 = 2$ Find the sum of all possible values of

$$\omega^4+\omega^3+\omega^2+\omega+1.$$

Solution:
The value of $\omega^4+\omega^3+\omega^2+\omega+1 = \frac{\omega^5-1}{\omega-1} = \frac{1}{\omega-1}$. The sum of these values is therefore the sum of $\frac{1}{\omega-1}$ over the five roots ω. Substituting $z=\omega+1$, we have that $(z-1)^5=2$, so $z^5+5z^4+10z^3+10z^2+5z-1=0$. The sum of the reciprocals of the roots of this equation is $-\frac{5}{-1} = \boxed{5}$ by Vieta's.

5. **IMO 1988 #4** Show that the solution set of the inequality
$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$
is a union of disjoint intervals, the sum of whose length is 1988.

Inequalities

*1. Let a, b, c be positive numbers. Prove that

$$(a+b)(b+c)(c+a) \geq 8abc$$

2. Let x, y, z be positive numbers. Prove that

$$xyz \geq (x+y-z)(y+z-x)(z+x-y)$$

3. Prove that for any non-negative real numbers a, b, c:

$$a^3+b^3+c^3+ab^2+bc^2+ca^2 \geq 2(a^2b+b^2c+c^2a)$$

*4. Which one is larger:

$$\sqrt[4]{5}+\sqrt[4]{8} \text{ or } \sqrt[4]{6}+\sqrt[4]{7}?$$

*5. Which one is larger:

$$17^{15} \text{ or } 31^{12}?$$

Solution:
$17^{15} > 16^{15} = (2^4)^{15} = 2^{60} = (2^5)^{12} = 32^{12} > 31^{12}$

6. Find the triples of positive real numbers (x, y, z) such that

$$\frac{2x + 2y + z}{\sqrt{x^2 + y^2 + z^2}} = 3.$$

7. Prove that for any three positive real numbers a_1, a_2, a_3,

$$\frac{a_1^2 + a_2^2 + a_3^2}{a_1^3 + a_2^3 + a_3^3} \geq \frac{a_1^3 + a_2^3 + a_3^3}{a_1^4 + a_2^4 + a_3^4}.$$

8. Let $f_1, f_2, ..., f_n$ be positive real numbers. Prove that for any real numbers $x_1, x_2, ..., x_n$, the quantity

$$f_1x_1^2 + f_2x_2^2 + \cdots + f_nx_n^2 - \frac{(f_1x_1 + f_2x_2 + \cdots + f_nx_n)^2}{f_1 + f_2 + \cdots + f_n}$$

is nonnegative.

9. **AIME II 2016 #15** For $1 \leq i \leq 215$ let $a_i = \dfrac{1}{2^i}$ and $a_{216} = \dfrac{1}{2^{215}}$. Let $x_1, x_2, ..., x_{216}$ be positive real numbers such that $\sum_{i=1}^{216} x_i = 1$ and $\sum_{1 \leq i < j \leq 216} x_i x_j = \dfrac{107}{215} + \sum_{i=1}^{216} \dfrac{a_i x_i^2}{2(1 - a_i)}$. The maximum possible value of $x_2 = \dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Sequences and Series

*1. Evaluate the following series:

(a) $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + ... = ?$

(b) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + ... + \dfrac{1}{99 \cdot 100} = ?$

(c) $\dfrac{1}{1 \cdot 3} + \dfrac{1}{2 \cdot 4} + \dfrac{1}{3 \cdot 5} + ... = ?$

Solution:

(a) This is a geometric series with common ratio $r = -\frac{1}{2}$. Thus, from our geometric series formula, we have

$$1 + (-\frac{1}{2})^1 + (-\frac{1}{2})^2 + (-\frac{1}{2})^3 + \cdots = \frac{1}{1 - (-1/2)} = \frac{1}{3/2} = \boxed{\frac{2}{3}}$$

.

(b) This series forms a telescoping series:

$$\begin{aligned}
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{99\cdot 100} &= \sum_{n=1}^{99}\frac{1}{n(n+1)}\\
&= \sum_{n=1}^{99}(\frac{1}{n}-\frac{1}{n+1})\\
&= (1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\cdots+(\frac{1}{99}-\frac{1}{100})\\
&= (1-\frac{1}{100}) = \boxed{\frac{99}{100}}
\end{aligned}$$

(c) This series forms a telescoping series:

$$\begin{aligned}
\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\ldots &= \sum_{n=1}^{\inf}\frac{1}{n(n+2)}\\
&= \sum_{n=1}^{\inf}(\frac{1/2}{n}-\frac{1/2}{n+2})\\
&= \frac{1}{2}[(1-\frac{1}{3})+(\frac{1}{2}-\frac{1}{4})+(\frac{1}{3}-\frac{1}{5})+(\frac{1}{4}-\frac{1}{6})+\ldots]\\
&= \frac{1}{2}(1+\frac{1}{2}) = \boxed{\frac{3}{4}}
\end{aligned}$$

2. In an arithmetic sequence of real numbers, the sum of the first two terms is 7 and the sum of the first six terms is 93. What is the sum of the first four terms?

3. **AMC 10A 2004 #24** Let $a_1, a_2, \ldots$ be a sequence with $a_1 = 1$ and $a_{2n} = n \cdot a_n$ for any positive integer n which is a power of 2. What is the value of $a_{2^{100}}$?

4. **AIME I 2012 #2** The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

5. **AIME 1985 #1** Let $x_1 = 97$, and for $n > 1$, let $x_n = \frac{n}{x_{n-1}}$. Calculate the product $x_1x_2x_3x_4x_5x_6x_7x_8$.

6. **AIME II 2005 #3** An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. Find the common ratio of the original series.

7. **AIME II 2008 #6** The sequence $\{a_n\}$ is defined by

$$a_0 = 1, a_1 = 1, \text{ and } a_n = a_{n-1} + \frac{a_{n-1}^2}{a_{n-2}} \text{ for } n \geq 2.$$

The sequence $\{b_n\}$ is defined by

$$b_0 = 1, b_1 = 3, \text{ and } b_n = b_{n-1} + \frac{b_{n-1}^2}{b_{n-2}} \text{ for } n \geq 2.$$

Find $\frac{b_{32}}{a_{32}}$.