Algebra Day 2 Notes

Math Circle Competition Team

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Source: "Complex Numbers" by Holden Lee

1 Intro to Complex Numbers

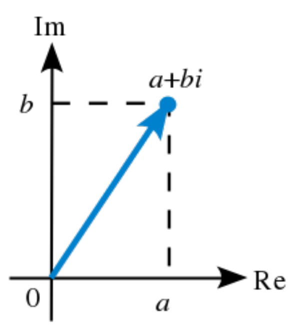
1.1 Definitions

The set of real numbers $\mathbb{R}$ is familiar to us and has a clear place in the world around us. But math is not quite complete without the complex numbers. In the reals, there is no solution to the polynomial expression

$$x^2 = -1,$$

as the square of any real number is positive. However, we can "create" a solution: let's add a number to $\mathbb{R}$ whose square is -1 and call it i. Since we want addition and multiplication to work out, we need to include all numbers of the form $z = a + bi$, where a and b are real. Call this new number system $\mathbb{C}$.

Definition 1.1. The set of numbers of the form $a + bi$, where a and b are real, is called the **complex numbers** and is denoted by $\mathbb{C}$. We call a the **real part** of z, denoted by $\text{Re}(z)$, and b the **imaginary part**, denoted by $\text{Im}(z)$.



A good way to imagine complex numbers is to think of them as points on a plane (the complex plane), by graphing $a + bi$ as the point (a, b). The x-axis consists of the real numbers $a \in \mathbb{R}$ (all a which are real numbers), while the y-axis consists of the imaginary numbers bi where $b \in \mathbb{R}$. However, the complex numbers have much more structure than just ordered pairs of real numbers - for instance, we can multiply complex numbers as shown below.

Let's see how basic operations work in $\mathbb{C}$. To add or subtract complex numbers, we simply add the real and imaginary components: $(1+2i)-(3-5i) = (1-3)i+(2-(-5))i = -2+7i$. We multiply complex numbers by the distributive law, and recall that $i^2 = -1$: for example, $(1 + 2i)(3 - 5i) = 1 \cdot 3 + 1 \cdot -5i + 2i \cdot 3 + 2i \cdot -5i = 3 - 5i + 6i - 10i^2 = 13 + i$. To define division, we need the following:

Definition 1.2. The conjugate of a complex number $z = a + bi$, where a and b are real, is $\overline{z} = a - bi$.

Note that the product of a complex number and its conjugate is always real:

$$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

This allows us to divide complex numbers: to evaluate $\dfrac{a + bi}{c + di}$ we multiply both the numerator and the denominator by the complex conjugate of $c + di$; that is, $c - di$. For example,

$$\frac{1 + 2i}{3 - 5i} = \frac{(1 + 2i)(3 + 5i)}{(3 - 5i)(3 + 5i)} = \frac{3 + 11i + 10i^2}{3^2 + 5^2} = -\frac{7}{34} + \frac{11}{34}i.$$

Note that complex conjugation preserves addition and multiplication; that is, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$. The **absolute value** or **modulus** of $z = a + bi$ is defined by

$$|z| = \sqrt{a^2 + b^2}.$$

It is the distance from the origin $(0, 0)$ to z when we plot z in the complex plane.

Now negative numbers have square roots too, since $(i\sqrt{a})^2 = -1$. With this, we can solve any quadratic equation by completing the square or using the quadratic formula. But what about general polynomial equations? Do we have to adjoin more elements to $\mathbb{C}$ so that every cubic polynomial $ax^3 + bx^2 + cx + d = 0$, where $a, b, c, d \in \mathbb{C}$, has a solution? Due to the following, we do not:

Theorem 1.1 (Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.*

This shows that $\mathbb{C}$ is a very "natural" number system to do algebra over.

1.2 Motivations

So what are complex numbers good for? Who cares about polynomial equations that can't be solved over the reals - after all, complex solutions don't make sense in real life, right?

Complex numbers were first developed to solve cubic polynomial equations - the formula hinged on the use of complex numbers even when the solutions of the equations were real. At first mathematicians were hesitant to adopt complex numbers, but complex numbers soon proved their use in a variety of other ways. A math problem (for example, a differential equation) might require as an intermediate step to solve a polynomial equation that may not have real solutions. But if we know how to work with complex numbers, we can proceed just as we would in the real case, and the simplified answer may in fact not have complex numbers at all, but would be hard to obtain otherwise.

Calculus over the complex numbers turns out to be much nicer - so that it is often advantageous to extend the domain of real functions to the complex numbers, and then look at their properties. This is the case with the infamous zeta function, which has applications to number theory.

Many problems that don't look like they involve complex numbers turn out to be much easier if we use them. For instance, to find Pythagorean triples we want to solve

$$a^2 + b^2 = c^2$$

over the integers (that is, where a, b, and c are integers). We may rewrite this as $c^2 - b^2 = a^2$ and factor as $(c - b)(c + b) = a^2$, and then use number theory. Suppose we want to do the same with

$$a^3 + b^3 = c^3.$$

(Think Fermat's Last Theorem!) We rewrite this as $c^3 - b^3 = a^3$ and factor

$$(c - b)(c^2 + cb + b^2) = a^3.$$

The left side is not completely factored (since there are still squared terms). If, however, we were working over the complex numbers, we could factor this further to:

$$(c - b)(c - \omega b)(c - \omega^2 b) = a^3$$

for some complex number ω (that's a lowercase "omega"). Then we could solve this by using number theory but over the complex numbers instead (this is part of what is known as algebraic number theory).

1.3 Examples

Let's warm up with some problems.

Example 1.1. Find the sum

$$1 + i + i^2 + \cdots + i^{2011}.$$

Solution. We calculate

$$1 = 1, \quad i = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1.$$

Hence the powers of i cycle through $1, i, -1$, and $-i$. So we group the sum into four terms at a time:

$$(1 + i + i^2 + i^3) + (i^4 + i^5 + i^6 + i^7) + \cdots + (i^{2008} + i^{2009} + i^{2010} + i^{2011})$$

and note that the sum of each group is $1 + i - 1 - i = 0$. So the answer is $\boxed{0}$.

Example 1.2. Show that if f is a polynomial with real coefficients and $f(z) = 0$, then $f(\overline{z}) = 0$. Conclude that the roots of f (with multiplicity) can be grouped into complex conjugate pairs.

Solution. Let $f(z) = a_n z^n + \cdots + a_1 z + a_0$. Using the fact that conjugation preserves addition and multiplication, we have

$$\begin{aligned} f(\overline{z}) &= a_n \overline{z}^n + \cdots + a_1 \overline{z} + a_0 \\ &= a_n \overline{z^n} + \cdots + a_1 \overline{z} + a_0 \\ &= \overline{a_n z^n} + \cdots + \overline{a_1 z} + \overline{a_0} \\ &= \overline{a_n z^n + \cdots + a_1 z + a_0} \\ &= \overline{f(z)}. \end{aligned}$$

Thus if $f(z) = 0$, then $f(\overline{z}) = 0$. Thus the nonreal roots of f come in complex conjugate pairs (that is, if $a + bi$ is a root of f, then so is $a - bi$). This is known as the **Complex Conjugate Root Theorem**.

Example 1.3 (AIME I 2009 #2)**.** There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

Find n.

Solution. We can write $z = a + 164i$, where a is real. Then

$$\frac{a + 164i}{(a+n) + 164i} = 4i.$$

Clearing the denominator gives

$$a + 164i = -656 + 4(a+n)i.$$

By matching real and imaginary parts of both sides of this equation, we have $a = -656$ and $164 = 4(a+n) = 4(-656+n)$. Solving for n yields $41 = -656 + n$ or $\boxed{n = 697}$.

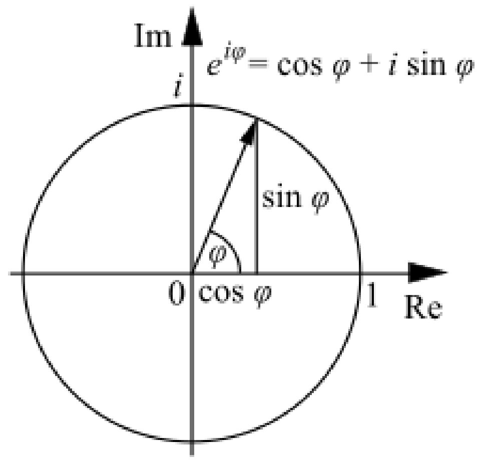
2 Polar Form, Euler, and De Moivre

2.1 Basic Facts and Examples

Consider the complex plane. When we write a complex number as $a + bi$, we graph the point by going horizontally a units and vertically b units. This is called the **rectangular** form of the complex number. Addition of complex numbers corresponds nicely to addition of vectors. Conjugation corresponds to reflection over the x-axis (real axis). But what do multiplication and division correspond to?

To answer this question, we need to write our complex numbers in a different form. We could instead specify a location z on the complex plane by the distance r from 0, and the angle θ made with the positive real axis. Drawing a triangle, we see that

$$z = r(\cos\theta + i\sin\theta)$$



which we abbreviate as $z = r\text{cis }\theta$. This is called the **polar** form, r is called the **modulus**, and θ is called the **argument**. Let's try to multiply complex numbers in this form. Let $z_1 = r_1\text{cis }\alpha$ and $z_2 = r_2\text{cis }\beta$. Then using the addition identities for sine and cosine,

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= r_1 r_2\left[(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \cos\beta\sin\alpha)\right] \\ &= r_1 r_2\left[\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right]. \end{aligned}$$

This is known as De Moivre's Theorem.

Theorem 2.1 (De Moivre)**.**

$$(r_1 cis\ \alpha)(r_2 cis\ \beta) = r_1 r_2 cis\ (\alpha+\beta).$$

In particular,

$$(r cis\ \theta)^n = r^n cis\ (n\theta).$$

Using this theorem we can find nth roots of a complex number z, that is, find the solutions to $x^n = z$. Writing $z = r\text{cis }\theta$ and $x = s\text{cis }\phi$, this is equivalent to $(s\text{cis }\phi)^n = r\text{cis }\theta$. Using De Moivre's Theorem, this is equivalent to

$$s^n\text{cis }(n\phi) = r\text{cis }\theta.$$

Hence we need $s^n = r$ and $n\phi = \theta$ as angles, so they are allowed to differ by a multiple of $2\pi i$. Hence $r = \sqrt[n]{s}$ and

$$\begin{aligned} n\phi &= \theta + 2\pi ik \\ \phi &= \frac{\theta}{n} + \frac{2\pi ik}{n} \end{aligned}$$

for some integer k. Taking $k = 0, 1, \dots, n-1$ (the n possible values of k modulo n) gives n distinct possible values of ϕ (other values differ from one of these by a multiple of $2\pi i$). Thus each nonzero complex number has n nth roots.

Example 2.1. Let $z \neq 0$ and $n > 1$. Show that the sum of the nth roots of z equals 0.

Solution 1. Let x be an nth root of z and $\omega = \operatorname{cis} \frac{2\pi}{n}$. Then $1, \omega, \omega^2, \dots, \omega^{n-1}$ are the nth roots of 1 so that $z, z\omega, z\omega^n, \dots, z\omega^{n-1}$ are the nth roots of z. So the sum is

$$z(1 + \omega + \dots \omega^{n-1}) = z \cdot \frac{\omega^n - 1}{\omega - 1} = 0$$

since $\omega^n = 1$.

Solution 2. Let $x_1, x_2, \dots, x_n$ be the nth roots of z and $\omega = \operatorname{cis} \frac{2\pi}{n}$ as above. Then $x_1\omega, \dots, x_n\omega$ are also the nth roots of unity, so are equal to $x_1, \dots, x_n$ in some order. Thus the sum s equals both $x_1 + \cdots + x_n$ and $\omega(x_1 + \cdots + x_n)$. Since $s = s\omega$, $s = 0$.

Example 2.2 (AIME II 2008 #9)**.** A particle is located on the coordinate plane at $(5, 0)$. Define a *move* for the particle as a counterclockwise rotation of $\frac{\pi}{4}$ radians about the origin followed by a translation of 10 units in the positive x-direction. Given that the particle's position after 150 moves is (p, q), find the greatest integer less than or equal to $|p| + |q|$.

Solution. Think of the coordinate plane as the complex plane, and let z_n be the particle's position after n moves, as a complex number. A counterclockwise rotation of $\frac{\pi}{4}$ corresponds to multiplication by $t = cis\frac{\pi}{4}$, and translation by 10 units in the positive x-direction corresponds to adding 10. Thus

$$z_n = tz_{n-1} + 10.$$

Calculating the first few terms,

$$\begin{aligned} z_0 &= 5 \\ z_1 &= 5t + 10 \\ z_2 &= (5t + 10)t + 10 = 5t^2 + 10t + 10 \end{aligned}$$

so we see that $z_n = 5t^n + 10t^{n-1} + \cdots + 10$, which could be proved by induction. Since t is an eighth root of 1, by Example 2.2 the sum of the eighth roots $1 + t + t^2 + \cdots + t^7$ equals 0. Since $t^8 = 1$, we have

$$0 = 1 + t + \cdots + t^7 = t^8 + \cdots + t^{15} = \cdots = t^{144} + \cdots + t^{151}.$$

Hence

$$\begin{aligned}
z_{150} &= 5t^{150} + 10t^{149} + \cdots + 10 \\
&= 10\left[(1+t+\cdots+t^7) + \cdots + (t^{144} + \cdots + t^{152})\right] - 10t^{151} - 5t^{150} \\
&= -10t^7 - 5t^6 \\
&= -10\text{cis}\,\frac{7\pi}{4} - 5\text{cis}\,\frac{3\pi}{2} \\
&= -10\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) + 5i.
\end{aligned}$$

Thus $|p| + |q| = 10\sqrt{2} + 5 \approx 10 \cdot 1.41 + 5 = 19.1$. The answer is $\boxed{19}$.

2.2 Trigonometry

The polar form of complex numbers can be used to simplify trig expressions and prove trig identities.

Example 2.3. Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Solution. By De Moivre's Theorem, we have

$$\begin{aligned}
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\
&= (\cos^3\theta - 3\cos\theta\sin^2\theta) + (3\cos^2\theta\sin\theta - \sin^3\theta)i.
\end{aligned}$$

Matching the real and imaginary parts,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta$$

and

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta.$$

Example 2.4. Show that $\cos 0^\circ + \cos 1^\circ + \cos 2^\circ + \cdots + \cos 89^\circ = \dfrac{1+\cot 0.5^\circ}{2}$.

Solution. Let $\omega = \cos 1^\circ + i\sin 1^\circ$. Then $\omega^n = \cos n^\circ + i\sin n^\circ$. Hence the desired sum equals the real part of $1 + \omega + \omega^2 + \cdots + \omega^{89}$. Using the geometric series formula,

$$
\begin{aligned}
1+\omega+\omega^2+\cdots+\omega^{89} = \frac{\omega^{90}-1}{\omega-1} &= \frac{i-1}{(\cos 1^\circ - 1) + i\sin 1^\circ} \\
&= \frac{(i-1)((\cos 1^\circ - 1) - i\sin 1^\circ)}{(\cos 1^\circ - 1)^2 + \sin^2 1^\circ} \\
&= \frac{(i-1)((\cos 1^\circ - 1) - i\sin 1^\circ)}{2 - 2\cos 1^\circ} \\
&= \frac{1 - \cos 1^\circ + \sin 1^\circ + (\cos 1^\circ - 1 + \sin 1^\circ)i}{2 - 2\cos 1^\circ}.
\end{aligned}
$$

This has real part

$$
\frac{1-\cos 1^\circ + \sin 1^\circ}{2 - 2\cos 1^\circ} = \frac{1}{2}\left(1 + \frac{\sin 1^\circ}{1 - \cos 1^\circ}\right) = \frac{1 + \cot 0.5^\circ}{2},
$$

where we used the trig identity $\cot\dfrac{\theta}{2} = \dfrac{\sin\theta}{1-\cos\theta}$.

3 Roots of Unity

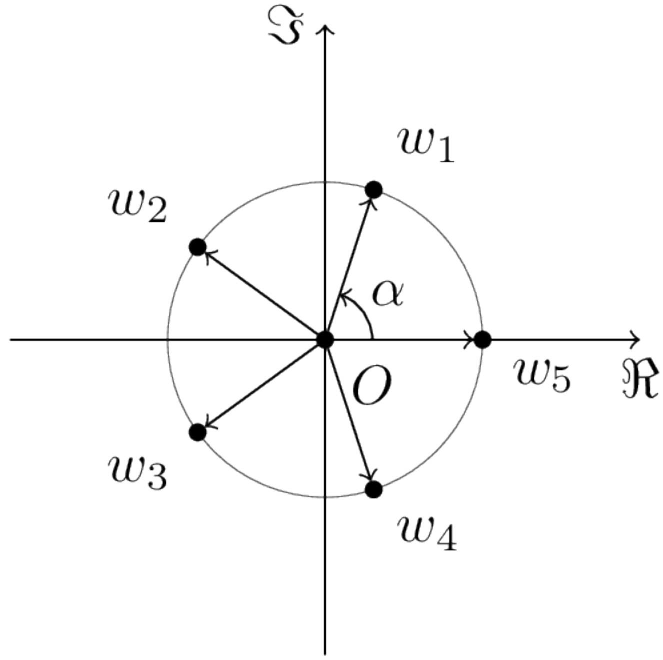
The nth roots of unity, i.e. the roots of 1, satisfy the equation

$$
x^n - 1 = 0.
$$

The nth roots of unity, not counting 1, satisfy

$$
\frac{x^n - 1}{x - 1} = x^{n-1} + \cdots + x + 1 = 0.
$$

If we plug in $x = \text{cis}\,\frac{2\pi}{n}$, the terms of this sum are all the nth roots of unity, and we get Example 2.2. This simple idea - the sum of the nth roots of unity satisfy the equations above, and that their sum is zero - can be very useful.



Example 3.1. n points $Q_1, \ldots, Q_n$ are equally spaced on a circle of radius 1 centered at O. Point P is on ray OQ_1 so that $OP = 2$. Find the product

$$\prod_{k=1}^{n} PQ_k$$

in closed form, in terms of n.

Solution. Letting ray OQ_1 be the positive real axis, the Q_i represent the nth roots of unity ω^i in the complex plane. Hence PQ_i equals $|2 - \omega^i|$. The roots of $x^n - 1 = 0$ are just the nth roots of unity, so $x^n - 1 = \prod_{i=0}^{n-1}(x - \omega^i)$. Plugging in $x = 2$ gives $\prod_{k=1}^{n} |PQ_i| = 2^n - 1$.