Algebra Day 2 Problem Set

Math Circle Competition Team

September 10th, 2017

1 Intro to Complex Numbers

1. * Find i^{4017}.

2. * If $c = 7 - 12i$, what is $\text{Re}(c) + \text{Im}(c) - \overline{c} + |c|^2$?

3. * **(AMC 12B 2004 #16)** A function f is defined by $f(z) = i\overline{z}$. How many values of z satisfy $|z| = 5$ and $f(z) = z$?

4. a and b are complex numbers such that

$$\frac{x}{a+b} = \frac{x}{a} + \frac{x}{b}$$

is true for all values of x. Find all possible values of $\frac{a}{b}$.

5. * **(AMC 12A 2007 #18)** The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2+i) = 0$. What is $a + b + c + d$?

6. **(AMC 12B 2008 #19)** A function f is defined by $f(z) = (4+i)z^2 + \alpha z + \gamma$ for all complex numbers z, where α and γ are complex numbers and $i^2 = -1$. Suppose that $f(1)$ and $f(i)$ are both real. What is the smallest possible value of $|\alpha| + |\gamma|$?

7. **(AIME I 2007 #3)** The complex number z is equal to $9 + bi$, where b is a positive real number. Given that the imaginary parts of z^2 and z^3 are equal, find b.

8. **(AIME I 2002 #12)** Let $F(z) = \frac{z+i}{z-i}$ for all complex numbers $z \neq i$, and let $z_n = F(z_{n-1})$ for all positive integers n. Given that $z_0 = \frac{1}{137} + i$ and $z_{2002} = a + bi$, where a and b are real, find $a + b$.

9. **(USAMO 1989 #3)** Let $P(z) = z^n + c_1 z^{n-1} + \cdots + c_n$ be a polynomial in z, with real coefficients c_k. Suppose that $|P(i)| < 1$. Prove that there exist real numbers a and b such that $P(a + bi) = 0$ and $(a^2 + b^2 + a)^2 < 4b^2 + 1$.

2 Polar Form, Euler, and De Moivre

1. Show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

2. * **(AMC 12B 2005 #22)** A sequence of complex numbers $z_0, z_1, \ldots$ is defined by the rule $z_{n+1} = \dfrac{iz_n}{\overline{z_n}}$. Suppose that $|z_0| = 1$ and $z_{2005} = 1$. How many possible values are there for z_0?

3. * **(AIME II 2000 #9)** Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3^\circ$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

4. **(AMC 12A 2008 #25)** A sequence (a_1, b_1), (a_2, b_2), (a_3, b_3), $\ldots$ of points in the coordinate plane satisfies

$$(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n).$$

   Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?

5. **(AIME II 2005 #9)** For how many positive integers n less than or equal to 1000 is

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

   true for all real t?

6. **(AIME 1994 #13)** The equation $x^{10} + (13x - 1)^{10} = 0$ has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

7. **(AIME II 2001 #14)** There are $2n$ complex numbers that satisfy both $z^{28} - z^8 - 1 = 0$ and $|z| = 1$. These numbers have the form $z_m = \cos\theta_m + i\sin\theta_m$, where $0 \le \theta_1 < \theta_2 < \ldots < \theta_{2n} < 360$ and angles are measured in degrees. Find the value of $\theta_2 + \theta_4 + \ldots + \theta_{2n}$.

3 Roots of Unity

1. Factor $x^5 + x + 1$.

2. **(AIME 1996 #11)** Let P be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have a positive imaginary part, and suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \le \theta < 360$. Find θ.

3. **(AIME 1997 #14)** Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let $\frac{m}{n}$ be the probability that $\sqrt{2 + \sqrt{3}} \le |v + w|$, where m and n are relatively prime positive integers. Find $m + n$.

4. **(AIME I 2004 #13)** The polynomial

$$P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$$

has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)], k = 1, 2, 3, \ldots, 34$, with $0 < a_1 \le a_2 \le a_3 \le \cdots \le a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

5. **(AIME II 2003 #15)** Let

$$P(x) = x + 2x^2 + 3x^3 \ldots 24x^{24} + 23x^{25} + 22x^{26} \ldots x^{47}.$$

Let $z_1, z_2, \ldots, z_r$ be the distinct zeros of $P(x)$, and let $z_k^2 = a_k + b_k i$ for $k = 1, 2, \ldots, r$, where a_k and b_k are real numbers. Let

$$\sum_{k=1}^{r} |b_k| = m + n\sqrt{p},$$

where m, n, and p are integers and p is not divisible by the square of any prime. Find $m + n + p$.