Algebra Day 2 Solutions

Math Circle Competition Team

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Intro to Complex Numbers

1. Find i^{4017}

   Solution

2. If $c = 7 - 12i$, what is $\text{Re}(c) + \text{Im}(c) - \overline{c} + |c|^2$?

   Solution
   We have $\text{Re}(c) = 7$, $\text{Im}(c) = -12$, $\overline{c} = 7 + 12i$, and $|c|^2 = (\sqrt{7^2 + (= 12)^2})^2 = 49 + 144 = 193$. Adding these together, remembering to add real parts to real parts and imaginary parts to imaginary parts, yields $7 - 12 - 7 - 12i + 193 = \boxed{181 - 12i}$.

3. **AMC 12B 2004 #16** A function f is defined by $f(z) = i\overline{z}$. How many values of z satisfy $|z| = 5$ and $f(z) = z$?

   Solution
   Let $z = a+bi$, so $\overline{z} = a-bi$. By definition, $z = a+bi = f(z) = i(a-bi) = b+ai$, which implies that all solutions to $f(z) = z$ lie on the line $y = x$ on the complex plane. The graph of $|z| = 5$ is a circle centered at the origin, and there are $2 \Rightarrow$ (C) intersections.

4. Let a and b are complex numbers such that

$$\frac{x}{a+b} = \frac{x}{a} + \frac{x}{b}$$

   is true for all values of x. Find all possible values of $\frac{a}{b}$.

   Solution

5. **AMC 12A 2007 #18** The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2+i) = 0$. What is $a + b + c + d$?

   Solution 1
   A fourth degree polynomial has four roots. Since the coefficients are real(meaning that complex roots come in conjugate pairs), the remaining two roots must be the complex conjugates of the two given roots, namely $2 - i, -2i$. Now we work backwards for the

polynomial: $(x-(2+i))(x-(2-i))(x-2i)(x+2i)=0$ $(x^2-4x+5)(x^2+4)=0$ $x^4-4x^3+9x^2-16x+20=0$ Thus our answer is $-4+9-16+20=9$.

Solution 2

Just like in Solution 1 we realize that the roots come in conjugate pairs. Which means the roots are $2i, i+2, -2i, 2-i$ So our polynomial is

(1) $f(x)=(x-2i)(x+2i)(x-i-2)(x-2+i)$

Looking at the equation of the polynomial $f(x)=x^4+ax^3+bx^2+cx+d$. We see that $a+b+c+d=f(1)-1$

If we plug in 1 into equation (1) we get $f(1)=(1-2i)(1+2i)(-1-i)(-1+i)$.

Now if we multiply a complex number by its conjugate we get the sum of the squares of its real and imaginary parts. Using this property on the above we multiply and get $f(1)=(1-2i)(1+2i)(-1-i)(-1+i)=(1^2+2^2)(1^2+1^2)=10$ So the answer is $f(1)-1=10-1=9$.

6. **AMC 12B 2008 #19** A function f is defined by $f(z)=(4+i)z^2+\alpha z+\gamma$ for all complex numbers z, where α and γ are complex numbers and $i^2=-1$. Suppose that $f(1)$ and $f(i)$ are both real. What is the smallest possible value of $|\alpha|+|\gamma|$?

   Solution

   We need only concern ourselves with the imaginary portions of $f(1)$ and $f(i)$ (both of which must be 0). These are:

$$\Im(f(1)) = i + i\Im(\alpha) + i\Im(\gamma)$$
$$\Im(f(i)) = -i + i\Re(\alpha) + i\Im(\gamma)$$

   Let $p=\Im(\gamma)$ and $q=\Re(\gamma)$, then we know $\Im(\alpha)=-p-1$ and $\Re(\alpha)=1-p$. Therefore

$$|\alpha|+|\gamma| = \sqrt{(1-p)^2+(-1-p)^2}+\sqrt{q^2+p^2} = \sqrt{2p^2+2}+\sqrt{p^2+q^2},$$

   which reaches its minimum $\sqrt{2}$ when $p=q=0$ by the Trivial Inequality. Thus, the answer is $\boxed{\sqrt{2}}$.

7. **AIME I 2007 #3** The complex number z is equal to $9+bi$, where b is a positive real number and $i^2=-1$. Given that the imaginary parts of z^2 and z^3 are the same, what is b equal to?

   Solution

   Squaring, we find that $(9+bi)^2=81+18bi-b^2$. Cubing and ignoring the real parts of the result, we find that $(81+18bi-b^2)(9+bi)=\ldots+(9\cdot 18+81)bi-b^3i$.

   Setting these two equal, we get that $18bi=243bi-b^3i$, so $b(b^2-225)=0$ and $b=-15,0,15$. Since $b>0$, the solution is $\boxed{015}$.

8. **AIME I 2002 #12** Let $F(z) = \dfrac{z+i}{z-i}$ for all complex numbers $z \neq i$, and let $z_n = F(z_{n-1})$ for all positive integers n. Given that $z_0 = \dfrac{1}{137} + i$ and $z_{2002} = a + bi$, where a and b are real numbers, find $a + b$.

Solution
Iterating F we get:

$$\begin{aligned}
F(z) &= \frac{z+i}{z-i} \\
F(F(z)) &= \frac{\frac{z+i}{z-i}+i}{\frac{z+i}{z-i}-i} = \frac{(z+i)+i(z-i)}{(z+i)-i(z-i)} = \frac{z+i+zi+1}{z+i-zi-1} = \frac{(z+1)(i+1)}{(z-1)(1-i)} \\
&= \frac{(z+1)(i+1)^2}{(z-1)(1^2+1^2)} = \frac{(z+1)(2i)}{(z-1)(2)} = \frac{z+1}{z-1}i \\
F(F(F(z))) &= \frac{\frac{z+1}{z-1}i+i}{\frac{z+1}{z-1}i-i} = \frac{\frac{z+1}{z-1}+1}{\frac{z+1}{z-1}-1} = \frac{(z+1)+(z-1)}{(z+1)-(z-1)} = \frac{2z}{2} = z.
\end{aligned}$$

From this, it follows that $z_{k+3} = z_k$, for all k. Thus $z_{2002} = z_{3\cdot 667+1} = z_1 = \frac{z_0+i}{z_0-i} = \frac{(\frac{1}{137}+i)+i}{(\frac{1}{137}+i)-i} = \frac{\frac{1}{137}+2i}{\frac{1}{137}} = 1 + 274i$.

Thus $a + b = 1 + 274 = \boxed{275}$.

9. **USAMO 1989 #3** Let $P(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n$ be a polynomial in the complex variable z, with real coefficients c_k. Suppose that $|P(i)| < 1$. Prove that there exist real numbers a and b such that $P(a+bi) = 0$ and $(a^2+b^2+1)^2 < 4b^2+1$.

Solution
Let $z_1, \ldots, z_n$ be the (not necessarily distinct) roots of P, so that

$$P(z) = \prod_{j=1}^{n} (z - z_j).$$

Since all the coefficients of P are real, it follows that if w is a root of P, then $P(\overline{w}) = \overline{P(w)} = 0$, so $\overline{w}$, the complex conjugate of w, is also a root of P.

Since

$$|i - z_1| \cdot |i - z_2| \cdots |i - z_n| = |P(i)| < 1,$$

it follows that for some (not necessarily distinct) conjugates z_i and z_j,

$$|z_i - i| \cdot |z_j - i| < 1.$$

Let $z_i = a + bi$ and $z_j = a - bi$, for real a, b. We note that

$$(a + b + 1)^2 - (a + b - 1)^2 = 4a + 4b.$$

Thus

$$\begin{aligned}(a^2 + b^2 + 1)^2 &= (a^2 + b^2 - 1)^2 + 4a^2 + 4b^2 = |a^2 + b^2 - 1 - 2ai|^2 + 4b^2 \\ &= |(a - i)^2 - (bi)^2|^2 + 4b^2 \\ &= \left(|a + bi - i| \cdot |a - bi - i|\right)^2 + 4b^2 \\ &= \left(|z_i - i| \cdot |z_j - i|\right)^2 + 4b^2 < 1 + 4b^2.\end{aligned}$$

Since $P(a + bi) = P(z_i) = 0$, these real numbers a, b satisfy the problem's conditions.
□

Polar Form, Euler, and De Moivre

1. Show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

.

Solution

2. **AMC 12B 2005 #22** A sequence of complex numbers $z_0, z_1, z_2, ...$ is defined by the rule

$$z_{n+1} = \frac{iz_n}{\overline{z_n}},$$

where $\overline{z_n}$ is the complex conjugate of z_n and $i^2 = -1$. Suppose that $|z_0| = 1$ and $z_{2005} = 1$. How many possible values are there for z_0?

Solution
Since $|z_0| = 1$, let $z_0 = e^{i\theta_0}$, where θ_0 is an argument of z_0. I will prove by induction that $z_n = e^{i\theta_n}$, where $\theta_n = 2^n(\theta_0 + \frac{\pi}{2}) - \frac{\pi}{2}$.

Base Case: trivial

Inductive Step: Suppose the formula is correct for z_k, then

$$z_{k+1} = \frac{iz_k}{\overline{z_k}} = ie^{i\theta_k}e^{i\theta_k} = e^{i(2\theta_k + \pi/2)}$$

Since
$$2\theta_k + \frac{\pi}{2} = 2 \cdot 2^n(\theta_0 + \frac{\pi}{2}) - \pi + \frac{\pi}{2} = 2^{n+1}(\theta_0 + \frac{\pi}{2}) - \frac{\pi}{2} = \theta_{n+1}$$
the formula is proven

$z_{2005} = 1 \Rightarrow \theta_{2005} = 2k\pi$, where k is an integer. Therefore,

$$2^{2005}(\theta_0 + \frac{\pi}{2}) = (2k + \frac{1}{2})\pi$$

$$\theta_0 = \frac{k}{2^{2004}}\pi + \left(\frac{1}{2^{2006}} - \frac{1}{2}\right)\pi$$

The value of θ_0 only matters modulo 2π. Since $\frac{k+2^{2005}}{2^{2004}}\pi \equiv \frac{k}{2^{2004}}\pi \mod 2\pi$, k only needs to take values from 0 to $2^{2005} - 1$, so the answer is $2^{2005} \Rightarrow \boxed{2^{2005}}$

3. **AIME II 2000 #9** Given that z is a complex number such that $z + \frac{1}{z} = 2\cos 3°$, find the least integer that is greater than $z^{2000} + \frac{1}{z^{2000}}$.

   Solution
   Using the quadratic equation on $z^2 - (2\cos 3)z + 1 = 0$, we have $z = \frac{2\cos 3 \pm \sqrt{4\cos^2 3 - 4}}{2} = \cos 3 \pm i \sin 3 = \text{cis}\, 3°$.

   There are other ways we can come to this conclusion. Note that if z is on the unit circle in the complex plane, then $z = e^{i\theta} = \cos\theta + i\sin\theta$ and $\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$. We have $z + \frac{1}{z} = 2\cos\theta = 2\cos 3°$ and $\theta = 3°$. Alternatively, we could let $z = a + bi$ and solve to get $z = \cos 3° + i\sin 3°$.

   Using De Moivre's Theorem we have $z^{2000} = \cos 6000° + i\sin 6000°$, $6000 = 16(360) + 240$, so $z^{2000} = \cos 240° + i\sin 240°$.

   We want $z^{2000} + \frac{1}{z^{2000}} = 2\cos 240° = -1$.

   Finally, the least integer greater than -1 is $\boxed{000}$.

4. **AMC 12A 2008 #25** A sequence (a_1, b_1), (a_2, b_2), (a_3, b_3), $\ldots$ of points in the coordinate plane satisfies
   $(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n)$ for $n = 1, 2, 3, \ldots$.
   Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?

   Solution
   This sequence can also be expressed using matrix multiplication as follows:
   $$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = 2 \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}.$$
   Thus, (a_{n+1}, b_{n+1}) is formed by rotating (a_n, b_n) counter-clockwise about the origin by $30°$ and dilating the point's position with respect to the origin by a factor of 2.

   So, starting with (a_{100}, b_{100}) and performing the above operations 99 times in reverse yields (a_1, b_1).

Rotating $(2,4)$ clockwise by $99 \cdot 30^\circ \equiv 90^\circ$ yields $(4,-2)$. A dilation by a factor of $\frac{1}{2^{99}}$ yields the point $(a_1, b_1) = \left(\frac{4}{2^{99}}, -\frac{2}{2^{99}}\right) = \left(\frac{1}{2^{97}}, -\frac{1}{2^{98}}\right)$.

Therefore, $a_1 + b_1 = \frac{1}{2^{97}} - \frac{1}{2^{98}} = \frac{1}{2^{98}} \Rightarrow D$.

Shortcut: no answer has 3 in the denominator. So the point cannot have orientation $(2,4)$ or $(-2,-4)$. Also there are no negative answers. Any other non-multiple of 90° rotation of $30n^\circ$ would result in the need of radicals. So either it has orientation $(4,-2)$ or $(-4,2)$. Both answers add up to 2. Thus, $2/2^{99} = \boxed{\frac{1}{2^{98}}}$.

5. **AIME II 2005 #9** For how many positive integers n less than or equal to 1000 is $(\sin t + i\cos t)^n = \sin nt + i\cos nt$ true for all real t?

Solution 1
We know by De Moivre's Theorem that $(\cos t + i\sin t)^n = \cos nt + i\sin nt$ for all real numbers t and all integers n. So, we'd like to somehow convert our given expression into a form from which we can apply De Moivre's Theorem.

Recall the trigonometric identities $\cos(\frac{\pi}{2} - u) = \sin u$ and $\sin(\frac{\pi}{2} - u) = \cos u$ hold for all real u. If our original equation holds for all t, it must certainly hold for $t = \frac{\pi}{2} - u$. Thus, the question is equivalent to asking for how many positive integers $n \le 1000$ we have that $\left(\sin\left(\frac{\pi}{2} - u\right) + i\cos\left(\frac{\pi}{2} - u\right)\right)^n = \sin n\left(\frac{\pi}{2} - u\right) + i\cos n\left(\frac{\pi}{2} - u\right)$ holds for all real u.

$\left(\sin\left(\frac{\pi}{2} - u\right) + i\cos\left(\frac{\pi}{2} - u\right)\right)^n = (\cos u + i\sin u)^n = \cos nu + i\sin nu$. We know that two complex numbers are equal if and only if both their real part and imaginary part are equal. Thus, we need to find all n such that $\cos nu = \sin n\left(\frac{\pi}{2} - u\right)$ and $\sin nu = \cos n\left(\frac{\pi}{2} - u\right)$ hold for all real u.

$\sin x = \cos y$ if and only if either $x + y = \frac{\pi}{2} + 2\pi \cdot k$ or $x - y = \frac{\pi}{2} + 2\pi \cdot k$ for some integer k. So from the equality of the real parts we need either $nu + n\left(\frac{\pi}{2} - u\right) = \frac{\pi}{2} + 2\pi \cdot k$, in which case $n = 1 + 4k$, or we need $-nu + n\left(\frac{\pi}{2} - u\right) = \frac{\pi}{2} + 2\pi \cdot k$, in which case n will depend on u and so the equation will not hold for all real values of u. Checking $n = 1 + 4k$ in the equation for the imaginary parts, we see that it works there as well, so exactly those values of n congruent to 1 (mod 4) work. There are $\boxed{250}$ of them in the given range.

Solution 2
This problem begs us to use the familiar identity $e^{it} = \cos(t) + i\sin(t)$. Notice, $\sin(t) + i\cos(t) = i(\cos(t) - i\sin(t)) = ie^{-it}$ since $\sin(-t) = -\sin(t)$. Using this, $(\sin(t) + i\cos(t))^n = \sin(nt) + i\cos(nt)$ is recast as $(ie^{-it})^n = ie^{-itn}$. Hence we must have $i^n = i \Rightarrow i^{n-1} = 1 \Rightarrow n \equiv 1 \bmod 4$. Thus since 1000 is a multiple of 4 exactly one quarter of the residues are congruent to 1 hence we have $\boxed{250}$.

Solution 3
We can rewrite $\sin(t)$ as $\cos\left(\frac{\pi}{2} - t\right)$ and $\cos(t)$ as $\sin\left(\frac{\pi}{2} - t\right)$. This means that $\sin t +$

$i\cos t = e^{i\left(\frac{\pi}{2}-t\right)} = \frac{e^{\frac{\pi i}{2}}}{e^{it}}$. This theorem also tells us that $e^{\frac{\pi i}{2}} = i$, so $\sin t + i\cos t = \frac{i}{e^{it}}$. By the same line of reasoning, we have $\sin nt + i\cos nt = \frac{i}{e^{int}}$.

For the statement in the question to be true, we must have $\left(\frac{i}{e^{it}}\right)^n = \frac{i}{e^{int}}$. The left hand side simplifies to $\frac{i^n}{e^{int}}$. We cancel the denominators and find that the only thing that needs to be true is that $i^n = i$. This is true if $n \equiv 1 \pmod 4$, and there are $\boxed{250}$ such numbers between 1 and 1000.

6. **AIME 1994 #13** The equation

$x^{10} + (13x-1)^{10} = 0$ has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}$.

Solution 1
Let $t = 1/x$. After multiplying the equation by t^{10}, $1+(13-t)^{10} = 0 \Rightarrow (13-t)^{10} = -1$.

Using DeMoivre, $13 - t = \text{cis}\left(\frac{(2k+1)\pi}{10}\right)$ where k is an integer between 0 and 9.

$t = 13 - \text{cis}\left(\frac{(2k+1)\pi}{10}\right) \Rightarrow \bar{t} = 13 - \text{cis}\left(-\frac{(2k+1)\pi}{10}\right)$.

Since $\text{cis}(\theta) + \text{cis}(-\theta) = 2\cos(\theta)$, $t\bar{t} = 170 - 26\cos\left(\frac{(2k+1)\pi}{10}\right)$ after expanding. Here k ranges from 0 to 4 because two angles which sum to 2π are involved in the product.

The expression to find is $\sum t\bar{t} = 850 - 26\sum_{k=0}^{4}\cos\frac{(2k+1)\pi}{10}$.

But $\cos\frac{\pi}{10} + \cos\frac{9\pi}{10} = \cos\frac{3\pi}{10} + \cos\frac{7\pi}{10} = \cos\frac{\pi}{2} = 0$ so the sum is $\boxed{850}$.

Solution 2
Divide both sides by x^{10} to get

$$1 + \left(13 - \frac{1}{x}\right)^{10} = 0$$

Rearranging:

$$\left(13 - \frac{1}{x}\right)^{10} = -1$$

Thus, $13 - \dfrac{1}{x} = \omega$ where $\omega = e^{i(\pi n/5 + \pi/10)}$ where n is an integer.

We see that $\dfrac{1}{x} = 13 - \omega$. Thus,

$$\frac{1}{x\overline{x}} = (13-\omega)(13-\overline{\omega}) = 169 - 13(\omega+\overline{\omega}) + \omega\overline{\omega} = 170 - 13(\omega+\overline{\omega})$$

Summing over all terms:

$$\frac{1}{r_1\overline{r_1}}+\cdots+\frac{1}{r_5\overline{r_5}}=5\cdot 170-13(e^{i\pi/10}+\cdots+e^{i(9\pi/5+\pi/10)})$$

However, note that $e^{i\pi/10}+\cdots+e^{i(9\pi/5+\pi/10)}=0$ from drawing the numbers on the complex plane, our answer is just

$$5\cdot 170=\boxed{850}$$

7. **AIME II 2001 #14** There are $2n$ complex numbers that satisfy both $z^{28}-z^8-1=0$ and $|\ z\ |=1$. These numbers have the form $z_m=\cos\theta_m+i\sin\theta_m$, where $0\le\theta_1<\theta_2<\ldots<\theta_{2n}<360$ and angles are measured in degrees. Find the value of $\theta_2+\theta_4+\ldots+\theta_{2n}$.

   Solution
   z can be written in the form $\operatorname{cis}\theta$. Rearranging, we find that $\operatorname{cis}28\theta=\operatorname{cis}8\theta+1$

   Since the real part of $\operatorname{cis}28\theta$ is one more than the real part of $\operatorname{cis}8\theta$ and their imaginary parts are equal, it is clear that either $\operatorname{cis}28\theta=\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $\operatorname{cis}8\theta=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, or $\operatorname{cis}28\theta=\frac{1}{2}-\frac{\sqrt{3}}{2}i$ and $\operatorname{cis}8\theta=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

   Case 1 : $\operatorname{cis}28\theta=\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $\operatorname{cis}8\theta=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ Setting up and solving equations, $Z^{28}=\operatorname{cis}60^\circ$ and $Z^8=\operatorname{cis}120^\circ$, we see that the solutions common to both equations have arguments $15^\circ,105^\circ,195^\circ$, and 285°

   Case 2 : $\operatorname{cis}28\theta=\frac{1}{2}-\frac{\sqrt{3}}{2}i$ and $\operatorname{cis}8\theta=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ Again setting up equations ($Z^{28}=\operatorname{cis}300^\circ$ and $Z^8=\operatorname{cis}240^\circ$) we see that the common solutions have arguments of $75^\circ,165^\circ,255^\circ$, and 345°

   Listing all of these values, we find that $\theta_2+\theta_4+\ldots+\theta_{2n}$ is equal to $(75+165+255+345)^\circ$ which is equal to $\boxed{840}$ degrees

Roots of Unity

1. Factor x^5+x+1.

2. **AIME 1996 #11** Let P be the product of the roots of $z^6+z^4+z^3+z^2+1=0$ that have a positive imaginary part, and suppose that $\mathrm{P}=r(\cos\theta^\circ+i\sin\theta^\circ)$, where $0<r$ and $0\le\theta<360$. Find θ.

   Solution 1

$$\begin{aligned}0&=z^6-z+z^4+z^3+z^2+z+1=z(z^5-1)+\frac{z^5-1}{z-1}\\0&=\frac{(z^5-1)(z(z-1)+1)}{z-1}=\frac{(z^2-z+1)(z^5-1)}{z-1}\end{aligned}$$

Thus $z^5 = 1, z \neq 1 \Longrightarrow z = \text{cis } 72, 144, 216, 288$,

or $z^2 - z + 1 = 0 \Longrightarrow z = \frac{1 \pm \sqrt{-3}}{2} = \text{cis } 60, 300$

Discarding the roots with negative imaginary parts (leaving us with $\text{cis}\theta,\ 0 < \theta < 180$), we are left with $\text{cis } 60, 72, 144$; their product is $P = \text{cis}(60 + 72 + 144) = \text{cis}\boxed{276}$.

Solution 2

Let w = the fifth roots of unity, except for 1. Then $w^6 + w^4 + w^3 + w^2 + 1 = w^4 + w^3 + w^2 + w + 1 = 0$, and since both sides have the fifth roots of unity as roots, we have $z^4 + z^3 + z^2 + z + 1 | z^6 + z^4 + z^3 + z^2 + 1$. Long division quickly gives the other factor to be $z^2 - z + 1$. The solution follows as above.

Solution 3

Divide through by z^3. We get the equation $z^3 + \frac{1}{z^3} + z + \frac{1}{z} + 1 = 0$. Let $x = z + \frac{1}{z}$. Then $z^3 + \frac{1}{z^3} = x^3 - 3x$. Our equation is then $x^3 - 3x + x + 1 = x^3 - 2x + 1 = (x-1)(x^2 + x - 1) = 0$, with solutions $x = 1, \frac{-1 \pm \sqrt{5}}{2}$. For $x = 1$, we get $z = \text{cis}60, \text{cis}300$. For $x = \frac{-1+\sqrt{5}}{2}$, we get $z = \text{cis}72, \text{cis}292$ (using exponential form of cos). For $x = \frac{-1-\sqrt{5}}{2}$, we get $z = \text{cis}144, \text{cis}216$. The ones with positive imaginary parts are ones where $0 \leq \theta \leq 180$, so we have $60 + 72 + 144 = \boxed{276}$.

3. **AIME 1997 #14** Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let $\frac{m}{n}$ be the probability that $\sqrt{2 + \sqrt{3}} \leq |v + w|$, where m and n are relatively prime positive integers. Find $m + n$.

Solution 1

$z^{1997} = 1 = 1(\cos 0 + i \sin 0)$ By De Moivre's Theorem, we find that $(k \in \{0, 1, \ldots, 1996\})$

$z = \cos\left(\frac{2\pi k}{1997}\right) + i \sin\left(\frac{2\pi k}{1997}\right)$ Now, let v be the root corresponding to $\theta = \frac{2\pi m}{1997}$, and let w be the root corresponding to $\theta = \frac{2\pi n}{1997}$. The magnitude of $v + w$ is therefore:

$$\sqrt{\left(\cos\left(\frac{2\pi m}{1997}\right) + \cos\left(\frac{2\pi n}{1997}\right)\right)^2 + \left(\sin\left(\frac{2\pi m}{1997}\right) + \sin\left(\frac{2\pi n}{1997}\right)\right)^2} = \sqrt{2 + 2\cos\left(\frac{2\pi m}{1997}\right)\cos\left(\frac{2\pi n}{1997}\right) + 2\sin\left(\frac{2\pi m}{1997}\right)\sin$$

We need $\cos\left(\frac{2\pi m}{1997}\right)\cos\left(\frac{2\pi n}{1997}\right) + \sin\left(\frac{2\pi m}{1997}\right)\sin\left(\frac{2\pi n}{1997}\right) \geq \frac{\sqrt{3}}{2}$. The cosine difference identity simplifies that to $\cos\left(\frac{2\pi m}{1997} - \frac{2\pi n}{1997}\right) \geq \frac{\sqrt{3}}{2}$. Thus, $|m - n| \leq \frac{\pi}{6} \cdot \frac{1997}{2\pi} = \lfloor \frac{1997}{12} \rfloor = 166$.

Therefore, m and n cannot be more than 166 away from each other. This means that for a given value of m, there are 332 values for n that satisfy the inequality; 166 of them $> m$, and 166 of them $< m$. Since m and n must be distinct, n can have 1996 possible values. Therefore, the probability is $\frac{332}{1996} = \frac{83}{499}$. The answer is then $499 + 83 = \boxed{582}$.

Solution 2

The solutions of the equation $z^{1997} = 1$ are the 1997th roots of unity and are equal to $\cos\left(\frac{2\pi k}{1997}\right) + i \sin\left(\frac{2\pi k}{1997}\right)$ for $k = 0, 1, \ldots, 1996$. They are also located at the vertices of a regular 1997-gon that is centered at the origin in the complex plane.

Without loss of generality, let $v = 1$. Then

$$\begin{aligned} |v+w|^2 &= |\cos\left(\frac{2\pi k}{1997}\right) + i\sin\left(\frac{2\pi k}{1997}\right) + 1|^2 \\ &= \left|\left[\cos\left(\frac{2\pi k}{1997}\right) + 1\right] + i\sin\left(\frac{2\pi k}{1997}\right)\right|^2 \\ &= \cos^2\left(\frac{2\pi k}{1997}\right) + 2\cos\left(\frac{2\pi k}{1997}\right) + 1 + \sin^2\left(\frac{2\pi k}{1997}\right) \\ &= 2 + 2\cos\left(\frac{2\pi k}{1997}\right) \end{aligned}$$

We want $|v+w|^2 \geq 2+\sqrt{3}$. From what we just obtained, this is equivalent to $\cos\left(\frac{2\pi k}{1997}\right) \geq \frac{\sqrt{3}}{2}$. This occurs when $\frac{\pi}{6} \geq \frac{2\pi k}{1997} \geq -\frac{\pi}{6}$ which is satisfied by $k = 166, 165, \ldots, -165, -166$ (we don't include 0 because that corresponds to v). So out of the 1996 possible k, 332 work. Thus, $m/n = 332/1996 = 83/499$. So our answer is $83 + 499 = \boxed{582}$.

Solution 3

We can solve a geometrical interpretation of this problem.

Without loss of generality, let $u = 1$. We are now looking for a point exactly one unit away from u such that the point is at least $\sqrt{2+\sqrt{3}}$ units away from the origin. Note that the "boundary" condition is when the point will be exactly $\sqrt{2+\sqrt{3}}$ units away from the origin; these points will be the intersections of the circle centered at $(1,0)$ with radius 1 and the circle centered at $(0,0)$ with radius $\sqrt{2+\sqrt{3}}$. The equations of these circles are $(x-1)^2 = 1$ and $x^2+y^2 = 2+\sqrt{3}$. Solving for x yields $x = \frac{\sqrt{3}}{2}$. Clearly, this means that the real part of v is greater than $\frac{\sqrt{3}}{2}$. Solving, we note that 332 possible vs exist, meaning that $\frac{m}{n} = \frac{332}{1996} = \frac{83}{499}$. Therefore, the answer is $83 + 499 = \boxed{582}$.

4. **AIME I 2004 #13** The polynomial $P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$ has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)], k = 1, 2, 3, \ldots, 34$, with $0 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

Solution

We see that the expression for the polynomial P is very difficult to work with directly, but there is one obvious transformation to make: sum the geometric series:

$$\begin{aligned} P(x) = \left(\frac{x^{18}-1}{x-1}\right)^2 - x^{17} &= \frac{x^{36} - 2x^{18} + 1}{x^2 - 2x + 1} - x^{17} \\ = \frac{x^{36} - x^{19} - x^{17} + 1}{(x-1)^2} &= \frac{(x^{19}-1)(x^{17}-1)}{(x-1)^2} \end{aligned}$$

This expression has roots at every 17th root and 19th roots of unity, other than 1. Since 17 and 19 are relatively prime, this means there are no duplicate roots. Thus, a_1, a_2, a_3, a_4 and a_5 are the five smallest fractions of the form $\frac{m}{19}$ or $\frac{n}{17}$ for $m, n > 0$.

$\frac{3}{17}$ and $\frac{4}{19}$ can both be seen to be larger than any of $\frac{1}{19}, \frac{2}{19}, \frac{3}{19}, \frac{1}{17}, \frac{2}{17}$, so these latter five are the numbers we want to add.

$\frac{1}{19} + \frac{2}{19} + \frac{3}{19} + \frac{1}{17} + \frac{2}{17} = \frac{6}{19} + \frac{3}{17} = \frac{6 \cdot 17 + 3 \cdot 19}{17 \cdot 19} = \frac{159}{323}$ and so the answer is $159 + 323 = \boxed{482}$.

5. **AIME II 2003 #15** Let $P(x) = x + 2x^2 + 3x^3 \ldots 24x^{24} + 23x^{25} + 22x^{26} \ldots x^{47}$. Let $z_1, z_2, \ldots, z_r$ be the distinct zeros of $P(x)$, and let $z_k^2 = a_k + b_k i$ for $k = 1, 2, \ldots, r$, where a_k and b_k are real numbers. Let

   $\sum_{k=1}^{r} |b_k| = m + n\sqrt{p}$, where m, n, and p are integers and p is not divisible by the square of any prime. Find $m + n + p$.

   Solution
   This can quite obviously be factored as:

$$P(x) = x\left(x^{23} + x^{22} + \cdots + x^2 + x + 1\right)^2$$

   Note that $(x^{23} + x^{22} + \cdots + x^2 + x + 1) \cdot (x - 1) = x^{24} - 1$. So the roots of $x^{23} + x^{22} + \cdots + x^2 + x + 1$ are exactly all 24-th complex roots of 1, except for the root $x = 1$.

   Let $\omega = \cos\frac{360^\circ}{24} + i \sin\frac{360^\circ}{24}$. Then the distinct zeros of P are $0, \omega, \omega^2, \ldots, \omega^{23}$.

   We can clearly ignore the root $x = 0$ as it does not contribute to the value that we need to compute.

   The squares of the other roots are $\omega^2,\ \omega^4,\ \ldots,\ \omega^{24} = 1,\ \omega^{26} = \omega^2,\ \ldots,\ \omega^{46} = \omega^{22}$.

   Hence we need to compute the following sum:

$$R = \sum_{k=1}^{23} \left| \sin\left(k \cdot \frac{360^\circ}{12}\right) \right|$$

   Using basic properties of the sine function, we can simplify this to

$$R = 4 \cdot \sum_{k=1}^{5} \sin\left(k \cdot \frac{360^\circ}{12}\right)$$

   The five-element sum is just $\sin 30^\circ + \sin 60^\circ + \sin 90^\circ + \sin 120^\circ + \sin 150^\circ$. We know that $\sin 30^\circ = \sin 150^\circ = \frac{1}{2}$, $\sin 60^\circ = \sin 120^\circ = \frac{\sqrt{3}}{2}$, and $\sin 90^\circ = 1$. Hence our sum evaluates to:

$$R = 4 \cdot \left(2 \cdot \frac{1}{2} + 2 \cdot \frac{\sqrt{3}}{2} + 1\right) = 8 + 4\sqrt{3}$$

Therefore the answer is $8+4+3=\boxed{015}$.

Solution 2

Note that $x^k + x^{k-1} + \cdots + x + 1 = \frac{x^{k+1}-1}{x-1}$. Our sum can be reformed as

$$\frac{x(x^{47}-1)+x^2(x^{45}-1)+\cdots+x^{24}(x-1)}{x-1}$$

So

$$\frac{x^{48}+x^{47}+x^{46}+\cdots+x^{25}-x^{24}-x^{23}-\cdots-x}{x-1}=0$$

$x(x^{47}+x^{46}+\cdots-x-1)=0$

$x^{47}+x^{46}+\cdots-x-1=0$

$x^{47}+x^{46}+\cdots+x+1=2(x^{23}+x^{22}+\cdots+x+1)$

$\frac{x^{48}-1}{x-1}=2\frac{x^{24}-1}{x-1}$

$x^{48}-1-2x^{24}+2=0$

$(x^{24}-1)^2=0$

And we can proceed as above.