# Combinatorics Notes

## Math Circle Competition Team

*September 17th, 2017*

**Source:** "Counting Strategies for Math Teams" by Chuck Garner

# 1 Intro to Combinatorics

## 1.1 The Basics

In general, counting can be very difficult. Here are some easy methods for solving counting problems:

**Addition.** There are $m$ varieties of soup and $n$ varieties of salad. So there are $m + n$ ways to eat soup or salad (but not both soup and salad).

**Multiplication.** The number of ways to eat both a soup and a salad is $m \cdot n$.

**Permutation.** There are $n$ distinct objects. The number of ways to reorder all of them is $n!$. The number of ways to reorder $r$ of them (where $r \leq n$) is

$$P(n, r) = \frac{n!}{(n-r)!}.$$

**The "Mississippi" Formula.** There are 11 letters in the word "MISSISSIPPI". The number of distinct orderings of those letters is

$$\frac{11!}{4!4!2!}.$$

In general, suppose we have a collection of $n$ objects that are indistinguishable except for their color. The number of objects of each color is $c_1, c_2, \ldots, c_k$. Then the number of distinct orderings of the marbles is

$$\frac{n!}{c_1!c_2!\ldots c_n!}.$$

**Combinations.** The number of ways to choose $k$ objects from a set of $n$ objects, when the order of the objects does not matter, is given by

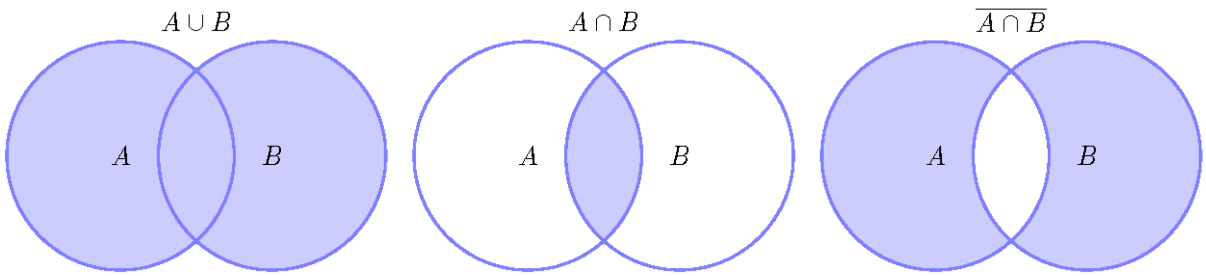
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. The expression $\binom{n}{k}$ is read as "n choose k" and is called a binomial coefficient.

## 1.2 Some Set Terminology

**Cardinality.** The *cardinality*, or size, of a set is the number of elements in the set.

**Union and Intersection.** The *union* of two sets $A$ and $B$, denoted $A \cup B$, is the set of all elements contained in either $A$ or $B$. The *intersection* of two sets $A$ and $B$, denoted $A \cap B$, is the set of elements contained in both $A$ and $B$.




**Complement.** The *complement* of a set $A$ with respect to a set $B$, denoted $A^C$ or $\overline{A}$, refers to all elements in $B$ that are not in $A$. For example, in the above image on the far right, $(A \cap B)^C$ (also denoted $\overline{A \cap B}$) contains those elements in $A$ or $B$ that are not in the intersection $A \cap B$.

**Example 1.1.** Define the sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Then the set $A = \{1, 2, 3\}$ has three elements, so its size is 3, denoted $|A| = 3$. Similarly, $|B| = 3$. Their union is $A \cup B = \{1, 2, 3, 4\}$, and their intersection is $A \cap B = \{2, 3\}$. $A^C = \{4\}$, since the only element in $B$ which is not in $A$ is 4.

# 2 Counting Techniques

## 2.1 Constructive Counting

The main goal of this technique is to determine how many integers/objects/etc. there are in a set by figuring out how to construct any element in the set. Throughout we keep track of the number of possibilities for each step, then perform the necessary manipulations in order to find the answer to the question.

**Example 2.1. (AIME II 2003 #3)** Define a *good word* as a sequence of letters that consists only of the letters $A$, $B$, and $C$ - some of these letters may not appear in the sequence - and in which $A$ is never immediately followed by $B$, $B$ is never immediately followed by $C$, and $C$ is never immediately followed by $A$. How many seven-letter good words are there?

*Solution.* There are three choices for the first letter in the word. For each letter that we choose, there are two possibilities for the next letter. (For example, if the first letter is an $A$, then the second letter can either be an $A$ or a $C$.) Similarly, there are two choices for the third letter, and so on. Thus, there are $3 \cdot 2^6 = \boxed{192}$ total seven-letter good words.

## 2.2 Stars and Bars

In how many ways can we distribute $n$ indistinguishable objects (represented below as stars) into $k$ bins? In the case where $n = 7$ and $k = 3$, we have seven stars:

$$\star \star \star \star \star \star \star$$

and we wish to separate them into 3 bins (represented by lines). For example, we might separate them as below:

$$\star \star \star \star \mid \star \mid \star \star$$

with one bin of 4 stars, one bin of 1 star, and one bin of 2 stars. We can put a separator in any gap between two stars (there are $n-1$ such gaps) and we will place $k-1$ separators, as this results in exactly $k$ bins. Thus the total number of ways to distribute $k$ objects into $n$ bins (when a bin must contain at least one object) is

$$\binom{n-1}{k-1}.$$

If, however, a bin can remain empty (contain 0 objects), the total number of ways is

$$\binom{n+k-1}{k-1}.$$

**Example 2.2.** How many ways can one distribute 8 indistinguishable one-dollar coins between 4 people so that each of them receives at least one dollar?

*Solution.* The 8 coins are our stars, and the 4 people are our bins. Thus one possible configuration might look like this:

$$* * \mid * \mid * * * \mid * *$$

which corresponds to Person 1 receiving 2 coins, Person 2 receiving 1 coin, Person 3 receiving 3 coins, and Person 4 receiving 2 coins. Since everyone must receive at least one coin (no bin is empty), the number of ways we can do this is $\binom{8-1}{4-1} = \binom{7}{3} = \boxed{35}$.

**Example 2.3.** How many solutions does the equation $a+b+c+d = 12$ have if $a$, $b$, $c$, and $d$ are nonnegative integers?

*Solution.* This is a clever use of Stars and Bars: we have $n = 12$ stars and wish to distribute them among $k = 4$ bins ($a, b, c$, and $d$). In this case, a bin may be empty, as the variables may be equal to 0. Thus the number of solutions is $\binom{12+4-1}{4-1} = \binom{15}{3} = \boxed{455}$.

## 2.3 Complementary Counting

Complementary Counting is a counting technique in which we count the complement of the set we want to count, then subtract that from the total number of possibilities. A key phrase in complementary counting problems is “at least”.

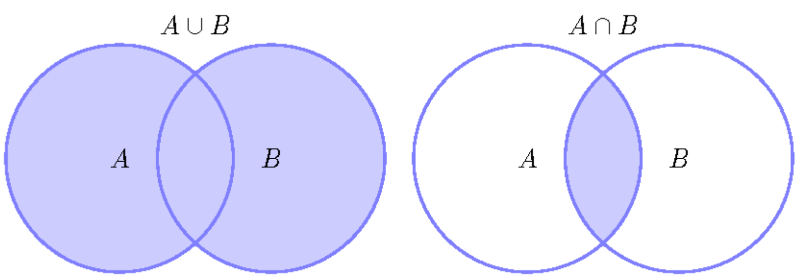
**Example 2.4. (AMC 10A 2006 #21)** How many four-digit positive integers have at least one digit that is a 2 or a 3?

*Solution.* We will use complementary counting to solve this problem; that is, we will find the number of four-digit positive integers that have neither a 2 nor a 3, and subtract this from the total number of possibilities. Our total number of four-digit positive integers is $9 \cdot 10 \cdot 10 \cdot 10 = 9000$, since we can pick any of 1 through 9 for the first digit and any of 0 through 9 for the second, third, and fourth. The number of four-digit positive integers with neither a 2 nor a 3 is $7 \cdot 8 \cdot 8 \cdot 8 = 3584$, since we can pick any of $1, 4, 5, 6, 7, 8, 9$ for the first digit and any of $0, 1, 4, 5, 6, 7, 8, 9$ for the second, third, and fourth. Thus our answer is $9000 - 3584 = \boxed{5416}$.

## 2.4 Inclusion-Exclusion Principle

The Inclusion-Exclusion Principle is a way of counting the elements in a union of two sets. In the below image, if we wanted to count the set $A \cup B$, we could add together the sizes of sets $A$ and $B$ and subtract the size of their intersection $A \cap B$ (since this intersection was added twice). Symbolically,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$




**Example 2.5. (AIME I 2002 #1)** Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Solution.* We will use the Inclusion-Exclusion Principle. First, if a license plate has an alphabetic palindrome, then whatever letter we choose as the first letter must also be the last letter; then we can have any 3 numbers following it. Thus the number of license plates with alphabetic palindromes is $26 \cdot 26 \cdot 1 \cdot 10 \cdot 10 \cdot 10$. Similarly, the number of numeric palindromes is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 1$. But we've counted the number of license plates with both alphabetic and numeric palindromes twice; there are $26 \cdot 26 \cdot 1 \cdot 10 \cdot 10 \cdot 1$ of these. Thus the number of license plates with palindromes is the sum of the first two minus the overlap, and the total number of plates is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$, so that the probability of having a palindromic license plate is

$$\frac{26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 + 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 - 26 \cdot 26 \cdot 10 \cdot 10}{26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10}$$

$$= \frac{1}{26} + \frac{1}{10} - \frac{1}{10 \cdot 26} = \frac{10 + 26 - 1}{10 \cdot 26} = \frac{35}{10 \cdot 26} = \frac{7}{52}.$$

So our answer is $52 + 7 = \boxed{59}$.