Combinatorics Problem Set

Math Circle Competition Team

September 17th, 2017

Intro to Combinatorics

$*$1. **(Mercer University 2002)** A typesetters apprentice, who is carrying a tray of letters forming the word MATHEMATICS, trips and spills the letters on the floor. If the apprentice randomly rearranges the letters into an eleven-letter word, what is the probability that the result will again be MATHEMATICS?

$*$2. **(The Art & Craft of Problem Solving)** We have three different toys and we want to give them away to two dogs and one cat (one toy per pet). The pets will be selected from four cats and six dogs. In how many ways can this be done?

$*$3. I have seven indistinguishable ping pong balls that are to be placed in 3 different boxes. In how many different ways may I fill the boxes? (Some boxes may be empty).

$*$4. How many positive integer solutions are there for the equation $x + y + z = 50$?

5. How many nonnegative solutions are there to the equation $x + 2y + 2z = 99$?

$*$6. A coin is flipped, a 6-sided die number 1 through 6 is rolled, and a 10-sided die numbered 0 through 9 is rolled. What is the probability that the coin comes up heads and the sum of the numbers that show on the dice is 8?

7. Particle Man is at the origin in three-dimensional space. How many ways can Particle Man take a series of 12 unit-length steps, each step parallel to one of the coordinate axes, from the origin to $(3, 4, 5)$ without passing through the point $(2, 3, 2)$?

$*$8. **(AoPS Intro to Counting & Probability)** How many 3-letter words can we make from the letters A, B, C, and D, if we are allowed to repeat letters, and we must use the letter A at least once?

$*$9. **(AMC 10A 2012)** A pair of six-sided fair dice are labeled so that one die has only even number (two each of 2, 4, and 6), and the other die has only odd numbers (two each of 1, 3, and 5). The pair of dice is rolled. What is the probability that the sum of the numbers on top of the two dice is 7?

10. **(Purple Comet HS 2013)** How many four-digit positive integers have exactly one digit equal to 1 and exactly one digit equal to 3?

11. **(AMC 10B 2013)** Let S be the set of sides and diagonals of a regular pentagon. A pair of elements of S are selected at random without replacement. What is the probability that the two chosen segments have the same length?

$*$12. **(AMC 10A 2013)** A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

$*$13. **(AMC 10A 2013)** A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?

$*$14. A restaurant has six appetizers, five main courses, and four desserts to choose from its menu. How many possible dinners are there if a main course is required but appetizers and desserts are not?

15. **(AMC 10A 2013)** How many three-digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?

16. **(AIME 2014)** Let the set $S = \{P_1, P_2, \dots P_{12}\}$ consist of the twelve vertices of a regular 12-gon. A subset Q of S is called communal if there is a circle such that all points of Q are inside the circle, and all points of S not in Q are outside of the circle. How many communal subsets are there? (Note that the empty set is a communal subset).

17. **(AMC 12A 2014)** A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

18. **(AMC 12A 2013)** Let S be the set $\{1, 2, 3, \dots, 19\}$. For $a, b \in S$, define $a \succ b$ to mean that either $0 < a - b \leq 9$ or $b - a > 9$. How many ordered triples (x, y, z) of elements of S have the property that $x \succ y$, $y \succ z$, and $z \succ x$?

19. How many ways are there to choose a 5-letter word from the 26-letter English alphabet with replacement, where words that are anagrams (rearrangements of each other) are considered the same?

20. **(2006 AMC 10A)** Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

21. **(2008 AMC 12B)** A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

22. **(2007 AIME II)** A mathematical organization is producing a set of commemorative license plates. Each plate contains a sequence of five characters chosen from the four letters in AIME and the four digits in 2007. No character may appear in a sequence more times than it appears among the four letters in AIME or the four digits in 2007. A set of plates in which each possible sequence appears exactly once contains N license plates. Find $\frac{N}{10}$.

23. **(2004 AIME I)** A convex polyhedron P has 26 vertices, 60 edges, and 36 faces, 24 of which are triangular and 12 of which are quadrilaterals. A space diagonal is a line segment connecting two non-adjacent vertices that do not belong to the same face. How many space diagonals does P have?