Combinatorics Day 1 Solutions

Math Circle Competition Team

September 17th, 2017

Intro to Combinatorics

*1. **Mercer University 2002** A typesetters apprentice, who is carrying a tray of letters forming the word MATHEMATICS, trips and spills the letters on the floor. If the apprentice randomly rearranges the letters into an eleven-letter word, what is the probability that the result will again be MATHEMATICS?

Solution

$\boxed{\frac{8}{11!}}$ There are $\frac{11!}{2!2!2!}$ ways to distinctly arrange the letters in "MATHEMATICS". Since this accounts for distinct words, our probability is $\frac{1}{\frac{11!}{8}} = \boxed{\frac{8}{11!}}$.

*2. **The Art & Craft of Problem Solving** We have three different toys and we want to give them away to two dogs and one cat (one toy per pet). The pets will be selected from four cats and six dogs. In how many ways can this be done?

Solution 1

Ignore order at first, but don't ignore type of animal. Then there are $\binom{4}{1}$ ways to pick the single cat and $\binom{6}{2}$ ways to pick the two dogs. Thus there are $\binom{4}{1} \cdot \binom{6}{2}$ ways of picking one cat and two dogs. But this does not distinguish between the order of individual pets (for example, "Ned, Minnie, Ruka" and "Ruka, Ned, Minnie"). In other words, we need to correct for order by multiplying by 3! (since we give away the toys, which are different, in order). So the answer is $\binom{4}{1} \cdot \binom{6}{2} \cdot 3! = \boxed{360}$.

Solution 2

Include order from the start. First, pick a cat (we can do this in $\binom{4}{1}$ ways). Then, pick a toy for this cat (three ways). Then pick the two dogs, but counting order ($P(6,2) = 6 \cdot 5$ ways). The answer is then $P(6,2) \cdot \binom{4}{1} \cdot 3 = \boxed{360}$.

*3. I have seven indistinguishable ping pong balls that are to be placed in 3 different boxes. In how many different ways may I fill the boxes? (Some boxes may be empty).

Solution

$\boxed{36}$ Use Stars and Bars. We have $n = 7$ objects and $k = 3$ bins, and we know that some bins can be empty. So the answer is $\binom{7+3-1}{3-1} = \binom{9}{2} = \boxed{36}$.

$*4.$ How many positive integer solutions are there for the equation $x + y + z = 50$?

Solution

$\boxed{1176}$ Use Stars and Bars. We can view this problem as having 50 objects and 3 bins, where none of the bins can be empty. So the answer is $\binom{50-1}{3-1} = \binom{49}{2} = \boxed{1176}$.

5. How many nonnegative solutions are there to the equation $x + 2y + 2z = 99$?

Solution

$\boxed{1275}$ Since 99 is odd and $2y + 2z$ is even, we know that x must be odd. We have a number of cases: if $x = 1$, $2(y + z) = 98$, so that $y + z = 49$. There are 50 ways this can be true. If $x = 3$, $2(y + z) = 96$, so that $y + z = 48$. There are 49 ways this can be true. We continue until the case $x = 99$, so that $y + z = 0$. There is 1 way this can be true. Thus the total number of solutions is $1 + 2 + \ldots + 49 + 50 = \frac{50 \cdot 51}{2} = \boxed{1275}$.

$*6.$ A coin is flipped, a 6-sided die number 1 through 6 is rolled, and a 10-sided die numbered 0 through 9 is rolled. What is the probability that the coin comes up heads and the sum of the numbers that show on the dice is 8?

Solution

$\boxed{1/20}$ The events are independent, so we can consider them separately and then multiply the probabilities. The probability that the coin comes up heads is $\frac{1}{2}$. There are 6 ways for the dice to add up to 8, and $6 \cdot 10$ total possible dice rolls, so that the probability that the dice add up to 8 is $\frac{6}{6 \cdot 10} = \frac{1}{10}$. Thus our answer is $\frac{1}{2}\frac{1}{10} = \boxed{\frac{1}{20}}$.

7. Particle Man is at the origin in three-dimensional space. How many ways can Particle Man take a series of 12 unit-length steps, each step parallel to one of the coordinate axes, from the origin to $(3, 4, 5)$ without passing through the point $(2, 3, 2)$?

Solution

$\boxed{23520}$ We use the Mississippi Formula. At first consider the total number of ways Particle Man can take a series of 12 unit-length steps from the origin to $(3, 4, 5)$, i.e. remove the constraint first. Consider a permutation of 3 identical x's, 4 identical y's, and 5 identical z's. For example, $xxxyyyyzzzzz$ is such a permutation. In each such permutation, an x corresponds to moving one step along the x-axis, a y corresponds to moving one step along the y-axis, and a z corresponds to moving one step along the z-axis. We are talking about moving positively along the axes here, because if a negative step is taken then the total number of steps taken to reach the destination will be more than 12. The total number of permutations is $\frac{12!}{3!4!5!}$ by the Mississippi Formula. Now we find the number of ways Particle Man can reach his destination passing through $(2, 3, 2)$. Consider the first part of his journey, i.e his journey from the origin to $(2, 3, 2)$. In a similar argument, Particle Man can go there in $\frac{7!}{2!2!3!}$ ways. Now consider the second part of his journey, i.e. his journey from $(2, 3, 2)$ to the destination

$(3, 4, 5)$. Similarly, this can be done in $\frac{5!}{3!1!1!}$ ways. Hence the total number of ways Particle Man can reach his destination via $(2, 3, 2)$ is $\frac{7!5!}{2!2!3!3!}$ (the numbers are multiplied). We have to subtract this from the total number of ways, so our final answer is $\dfrac{12!}{3!4!5!} - \dfrac{7!5!}{2!2!3!3!} = \boxed{23520}$.

∗8. **AoPS Intro to Counting & Probability** How many 3-letter words can we make from the letters A, B, C, and D, if we are allowed to repeat letters, and we must use the letter A at least once?

Solution

$\boxed{37}$ We have 3 cases: our word contains 1 A, our word contains 2 A's, or our word contains 3 A's. If our word contains 3 A's, there is only 1 possible word. If our word contains 2 A's, there are 3 possibilities for the third letter and $\binom{3}{2} = 3$ ways to order our 2 A's, for a total of $3 \cdot 3 = 9$ words. Finally, if our word contains 1 A, we have 3 choices for the second letter, 3 choices for the third letter, and $\binom{3}{1} = 3$ choices for the position of our A, for a total of $3 \cdot 3 \cdot 3 = 27$ words. Thus our total is $27 + 9 + 1 = \boxed{37}$.

∗9. **AMC 10A 2012** A pair of six-sided fair dice are labeled so that one die has only even number (two each of 2, 4, and 6), and the other die has only odd numbers (two each of 1, 3, and 5). The pair of dice is rolled. What is the probability that the sum of the numbers on top of the two dice is 7?

Solution 1

To solve this, we need to find the number of ways that we can roll a sum of 7 divided by the total possible rolls.

The total number of combinations when rolling two dice is $6 * 6 = 36$.

There are three ways that a sum of 7 can be rolled. $2 + 5$, $4 + 3$, and $6 + 1$. There are two 2's on one die and two 5's on the other, so there are a total of 4 ways to roll the combination of 2 and 5. There are two 4's on one die and two 3's on the other, so there are a total of 4 ways to roll the combination of 4 and 3. There are two 6's on one die and two 1's on the other, so there are a total of 4 ways to roll the combination of 6 and 1. Add $4 + 4 + 4 = 12$.

Thus, our probability is $\frac{12}{36} = \frac{1}{3}$. The answer is $\boxed{\dfrac{1}{3}}$.

Solution 2

Assume we roll the die with only evens first. For whatever value rolled, there are exactly 2 faces on the odd die that makes the sum 7. The odd die has 6 faces, so our probability is $\boxed{\dfrac{1}{3}}$.

10. **Purple Comet HS 2013** How many four-digit positive integers have exactly one digit equal to 1 and exactly one digit equal to 3?

Solution
$\boxed{720}$

11. **AMC 10B 2013** Let S be the set of sides and diagonals of a regular pentagon. A pair of elements of S are selected at random without replacement. What is the probability that the two chosen segments have the same length?

Solution
$\boxed{4/9}$ In a regular pentagon, there are 5 sides with the same length, and 5 diagonals with the same length. Picking an element at random will leave 4 elements with the same length as the element picked, with 9 total elements remaining. Therefore, the probability is $\boxed{\frac{4}{9}}$.

∗12. **AMC 10A 2013** A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

Solution
$\boxed{10}$ Let the number of students on the council be x. We know that there are $\binom{x}{2}$ ways to choose a two person team. This gives that $x(x-1) = 20$, which has a positive integer solution of 5. If there are 5 people on the welcoming committee, then there are $\binom{5}{3} = \boxed{10}$ ways to choose a three-person committee.

∗13. **AMC 10A 2013** A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?

Solution 1
Let us split this up into two cases.

Case 1: The student chooses both algebra and geometry.

This means that 3 courses have already been chosen. We have 3 more options for the last course, so there are 3 possibilities here.

Case 2: The student chooses one or the other.

Here, we simply count how many ways we can do one, multiply by 2, and then add to the previous.

Assume the mathematics course is algebra. This means that we can choose 2 of History, Art, and Latin, which is simply $\binom{3}{2} = 3$. If it is geometry, we have another 3 options, so we have a total of 6 options if only one mathematics course is chosen.

Thus, overall, we can choose a program in $6 + 3 = \boxed{9}$ ways

Solution 2
We can use complementary counting. Since there must be an English class, we will add that to our list of classes for 3 remaining spots for the classes. We are also told that there needs to be at least one math class. This calls for complementary counting. The total number of ways of choosing 3 classes out of the 5 is $\binom{5}{3}$. The total number of ways of choosing only non-mathematical classes is $\binom{3}{3}$. Therefore the amount of ways you can pick classes with at least one math class is $\binom{5}{3} - \binom{3}{3} = 10 - 1 = \boxed{9}$ ways.

∗14. A restaurant has six appetizers, five main courses, and four desserts to choose from its menu. How many possible dinners are there if a main course is required but appetizers and desserts are not?

Solution
$\boxed{175}$ We consider "no appetizer" to be another choice for appetizer and "no dessert" to be another choice for dessert. Thus there are 7 choices of appetizer, 5 choices of main course, and 5 choices of dessert, for a total of $7 \cdot 5 \cdot 5 = \boxed{175}$ possible dinners.

15. **AMC 10A 2013** How many three-digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?

Solution
$\boxed{60}$ These three digit numbers are of the form xyx. We see that $x \neq 0$ and $x \neq 5$, as $x = 0$ does not yield a three-digit integer and $x = 5$ yields a number divisible by 5.

The second condition is that the sum $2x + y < 20$. When x is 1, 2, 3, or 4, y can be any digit from 0 to 9, as $2x < 10$. This yields $10(4) = 40$ numbers.

When $x = 6$, we see that $12 + y < 20$ so $y < 8$. This yields 8 more numbers.

When $x = 7$, $14 + y < 20$ so $y < 6$. This yields 6 more numbers.

When $x = 8$, $16 + y < 20$ so $y < 4$. This yields 4 more numbers.

When $x = 9$, $18 + y < 20$ so $y < 2$. This yields 2 more numbers.

Summing, we get $40 + 8 + 6 + 4 + 2 = \boxed{60}$.

16. **AIME 2014** Let the set $S = \{P_1, P_2, \ldots P_{12}\}$ consist of the twelve vertices of a regular 12-gon. A subset Q of S is called communal if there is a circle such that all points of Q are inside the circle, and all points of S not in Q are outside of the circle. How many communal subsets are there? (Note that the empty set is a communal subset).

Solution

$\boxed{134}$ By looking at the problem and drawing a few pictures, we see that one cannot draw a circle that covers 2 disjoint areas of the 12-gon without including all the vertices in between those areas. In other words, in order for a subset to be communal, all the vertices in the subset must be adjacent to one another. We now count the number of ways to select a row of adjacent vertices. We notice that for any subset size between 1 and 11, there are 12 possible subsets like this (this is true because we can pick any of the 12 vertices as a "starting" vertex, include some number of vertices counterclockwise from that vertex, and generate all possible configurations). However, we also have to include the set of all 12 vertices, as well as the empty set. Thus, the total number is $12 * 11 + 2 = \boxed{134}$.

17. **AMC 12A 2014** A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

Solution 1

We can discern three cases.

Case 1: Each room houses one guest. In this case, we have 5 guests to choose for the first room, 4 for the second, ..., for a total of $5! = 120$ assignments.

Case 2: Three rooms house one guest; one houses two. We have $\binom{5}{3}$ ways to choose the three rooms with 1 guest, and $\binom{2}{1}$ to choose the remaining one with 2. There are $5 \cdot 4 \cdot 3$ ways to place guests in the first three rooms, with the last two residing in the two-person room, for a total of $\binom{5}{3}\binom{2}{1} \cdot 5 \cdot 4 \cdot 3 = 1200$ ways.

Case 3: Two rooms house two guests; one houses one. We have $\binom{5}{2}$ to choose the two rooms with two people, and $\binom{3}{1}$ to choose one remaining room for one person. Then there are 5 choices for the lonely person, and $\binom{4}{2}$ for the two in the first two-person room. The last two will stay in the other two-room, so there are $\binom{5}{2}\binom{3}{1} \cdot 5 \cdot \binom{4}{2} = 900$ ways.

In total, there are $120 + 1200 + 900 = \boxed{2220}$ assignments.

Solution 2

We can work in reverse by first determining the number of combinations in which there are more than 2 friends in at least one room. There are three cases:

Case 1: Three friends are in one room. Since there are 5 possible rooms in which this can occur, we are choosing three friends from the five, and the other two friends can each be in any of the four remaining rooms, there are $5 \cdot \binom{5}{3} \cdot 4 \cdot 4 = 800$ possibilities.

Case 2: Four friends are in one room. Again, there are 5 possible rooms, we are choosing four of the five friends, and the other one can be in any of the other four rooms, so there are $5 \cdot \binom{5}{4} \cdot 4 = 100$ possibilities.

Case 3: Five friends are in one room. There are 5 possible rooms in which this can occur, so there are 5 possibilities.

Since there are $5^5 = 3125$ possible combinations of the friends, the number fitting the given criteria is $3125 - (800 + 100 + 5) = \boxed{2220}$.

18. **AMC 12A 2013** Let S be the set $\{1, 2, 3, \ldots, 19\}$. For $a, b \in S$, define $a \succ b$ to mean that either $0 < a - b \leq 9$ or $b - a > 9$. How many ordered triples (x, y, z) of elements of S have the property that $x \succ y$, $y \succ z$, and $z \succ x$?

Solution

$\boxed{855}$ Imagine 19 numbers are just 19 persons sitting evenly around a circle C; each of them is facing to the center.

One may check that $x \succ y$ if and only if y is one of the 9 persons on the left of x, and $y \succ x$ if and only if y is one of the 9 persons on the right of x. Therefore, "$x \succ y$ and $y \succ z$ and $z \succ x$" implies that x, y, z cuts the circumference of C into three arcs, each of which has no more than 10 numbers sitting on it (inclusive).

We count the complement: where the cut generated by (x, y, z) has ONE arc that has more than 10 persons sitting on. Note that there can only be one such arc because there are only 19 persons in total.

Suppose the number of persons on the longest arc is $k > 10$. Then two places of x, y, z are just chosen from the two end-points of the arc, and there are $19 - k$ possible places for the third person. Once the three places of x, y, z are chosen, there are three possible ways to put x, y, z into them clockwise. Also, note that for any $k > 10$, there are 19 ways to choose an arc of length k. Therefore the total number of ways (of the complement) is

$$\sum_{k=11}^{18} 3 \cdot 19 \cdot (19 - k) = 3 \cdot 19 \cdot (1 + \cdots + 8) = 3 \cdot 19 \cdot 36$$

So the answer is

$$3 \cdot \binom{19}{3} - 3 \cdot 19 \cdot 36 = 3 \cdot 19 \cdot (51 - 36) = \boxed{855}$$

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19. How many ways are there to choose a 5-letter word from the 26-letter English alphabet with replacement, where words that are anagrams (rearrangements of each other) are considered the same?

Solution

$\boxed{142506}$ Since anagrams are the same (for example, *aaaab* and *baaaa* are the same), what matters is how many times each letter appears in the word. If we let C_a be the number of times a appears, C_b be the number of times b appears, and so on, then we have

$$C_a + C_b + C_c + \ldots + C_y + C_z = 5,$$

as there are 5 letters in the word. This is another clever use of Stars and Bars: we have $n = 5$ stars and wish to distribute them among $k = 26$ bins (which may be empty). The number of ways to do this is $\binom{5+26-1}{26-1} = \binom{30}{25} = \boxed{142506}$.

20. **(2006 AMC 10A)** Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

Solution

$\boxed{1}$ Look at the remainders of these integers mod 5. Our only possibilities for remainders mod 5 are 0, 1, 2, 3, or 4; if we choose a set of six integers, then by the Pigeonhole Principle, there must be two integers in our set which have the same remainder mod 5. Call these x and y. Then $x - y \equiv 0 \pmod 5$, and thus $x - y$ is divisible by 5. Since two such integers will exist in any set of 6 integers (by Pigeonhole, above), the probability is 100% or $\boxed{1}$.

21. **(2008 AMC 12B)** A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

Solution

$\boxed{\frac{17}{28}}$ We will use complementary counting; that is, we want to count the number of ways that Auntie Em would NOT be able to park. This can only happen if there are no 2 adjacent spaces. That means that following any open space, we must have a parked car; in other words, once we pick a position for an open space, the next space is fixed as a parked car. This fixes 3 of our 16 spaces, so that we only have 13 choices for the empty spaces. The number of possible arrangements is then $\binom{13}{4}$. There are $\binom{16}{4}$ total possible arrangements of empty spaces, so the probability that the cars are arranged so that there are no 2 adjacent spaces is thus $\frac{\binom{13}{4}}{\binom{16}{4}} = \frac{\frac{13\cdot12\cdot11\cdot10}{4\cdot3\cdot2\cdot1}}{\frac{16\cdot15\cdot14\cdot13}{4\cdot3\cdot2\cdot1}} = \frac{13\cdot12\cdot11\cdot10}{16\cdot15\cdot14\cdot13} = \frac{11}{28}$. But this was the probability of not being able to park, so the probability that she can park is $1 - \frac{11}{28} = \boxed{\frac{17}{28}}$.

22. **(2007 AIME II)** A mathematical organization is producing a set of commemorative license plates. Each plate contains a sequence of five characters chosen from the four letters in AIME and the four digits in 2007. No character may appear in a sequence more times than it appears among the four letters in AIME or the four digits in 2007. A set of plates in which each possible sequence appears exactly once contains N license plates. Find $\frac{N}{10}$.

    Solution
    $\boxed{372}$ We consider two cases: the case where a 0 appears one time or not at all, and the case where a 0 appears twice. If the 0 appears once or not at all, then we have $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$ possible choices for our license plate. If a 0 appears twice, then we have $\binom{5}{2} = 10$ possible places for our 0's (since the order of the 0's doesn't matter), and we may then place our 6 remaining characters into the final three places. Thus we have $10 \cdot 6 \cdot 5 \cdot 4 = 1200$ possible license plates. Our total is then $2520 + 1200 = 3720$, and since the problem asks for this number divided by 10, our answer is $\boxed{372}$.

23. **(2004 AIME I)** A convex polyhedron P has 26 vertices, 60 edges, and 36 faces, 24 of which are triangular and 12 of which are quadrilaterals. A space diagonal is a line segment connecting two non-adjacent vertices that do not belong to the same face. How many space diagonals does P have?

    Solution
    $\boxed{241}$ If we pick any two vertices, we have three possibilities for the line between them: it is either an edge of a face, a diagonal of a face, or a space diagonal. We have 60 edges, and diagonals of faces occur exactly twice per quadrilateral, for a total of 24. But the total number of ways to pick two vertices is $\binom{26}{2} = 325$, so the number of space diagonals is $325 - 60 - 24 = \boxed{241}$.