Probability Notes

Math Circle Competition Team

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1 Combinatorics Recap

The "Mississippi" Formula. There are 11 letters in the word "MISSISSIPPI". The number of distinct orderings of those letters is

$$\frac{11!}{4!4!2!}.$$

In general, suppose we have a collection of n objects that are indistinguishable except for their color. The number of objects of each color is $c_1, c_2, \ldots, c_k$. Then the number of distinct orderings of the marbles is

$$\frac{n!}{c_1!c_2!\ldots c_n!}.$$

Combinations. The number of ways to choose k objects from a set of n objects, when the order of the objects does not matter, is given by

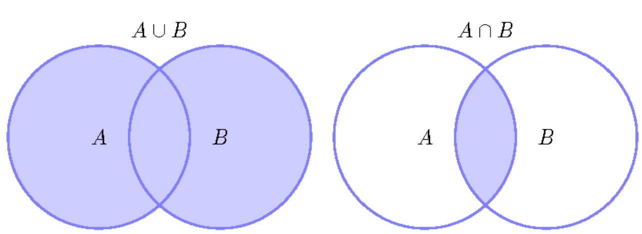
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. The expression $\binom{n}{k}$ is read as "n choose k" and is called a binomial coefficient.

Complementary Counting. Complementary Counting is a counting technique in which we count the complement of the set we want to count, then subtract that from the total number of possibilities. A key phrase in complementary counting problems is "at least". This technique can also be used in probability problems.

Inclusion-Exclusion Principle. The Inclusion-Exclusion Principle is a way of counting the elements in a union of two sets. In the below image, if we wanted to count the set $A \cup B$, we could add together the sizes of sets A and B and subtract the size of their intersection $A \cap B$ (since this intersection was added twice). Symbolically,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$



2 Probability

2.1 Casework

Casework suggests that sometimes, there are multiple distinct cases that behave in their own special ways, and in order to consider all possibilities we have to divide the calculation into several smaller pieces.

Example 2.1. (AIME I 2014 #2) An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N.

Solution. We do casework based on which color is picked.

- *CASE 1: Both marbles are green.* Then the requested probability is

$$\frac{4}{4+6} \cdot \frac{16}{N+16} = \frac{32}{5(N+16)}.$$

- *CASE 2: Both marbles are blue.* Then the requested probability is

$$\frac{6}{4+6} \cdot \frac{N}{N+16} = \frac{3N}{5(N+16)}.$$

Adding both cases and setting the resulting expression equal to the desired probability gives the equation

$$\frac{32+3N}{5(N+16)} = \frac{58}{100}.$$

Solving for N yields $N = \boxed{144}$.

2.2 Binomial Coefficients

Binomial Coefficients. The number of different ways to pick k objects from a set of n objects (when the order of the objects **does not** matter) is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $\binom{n}{k}$ is read as "n choose k". This is known as a *binomial coefficient*, and was previously referred to as a combination.

Example 2.2. A five-card hand is drawn from a standard deck of 52 cards.

1. How many five-card hands are possible?
   Solution. Since the order of the cards in the hand does not matter the number of hands is $\binom{52}{5} = \frac{52!}{5!47!}$. Note: you do not have to simplify, unless specifically stated.

2. What is the probability that all the cards in the hand drawn are hearts?
Solution. There are 13 hearts in the deck, so the number of 5 cards hands with all hearts is $\binom{13}{5}$. Thus the probability of getting all hearts is $\dfrac{\binom{13}{5}}{\binom{52}{5}} = \dfrac{\frac{13!}{5!8!}}{\frac{52!}{5!47!}} = \dfrac{13! \cdot 5! \cdot 47!}{5! \cdot 8! \cdot 52!}$

Basic Identities. Binomial coefficients obey the following basic identities:

- $\binom{n}{k} = \binom{n}{n-k}$ (*Symmetry*) $\qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ (*Recursion*)
- $\binom{n}{i}\binom{i}{j} = \binom{n}{j}\binom{n-j}{i-j}$ (*Trinomial Revision*)

Pascal's Triangle. When the binomial coefficients are arranged in a triangle, they form what is known as *Pascal's triangle*:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array} \qquad \begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \end{array}$$

where the right-hand triangle shows the binary coefficient representations. Each number in Pascal's triangle is the sum of the two numbers above it.

Sigma Notation. We sometimes denote sums by expressions like $\sum_{k=0}^{n} a_k$. The large symbol is a capital sigma, and it represents the sum of the terms a_0, a_1, ..., a_n:

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + \ldots + a_n.$$

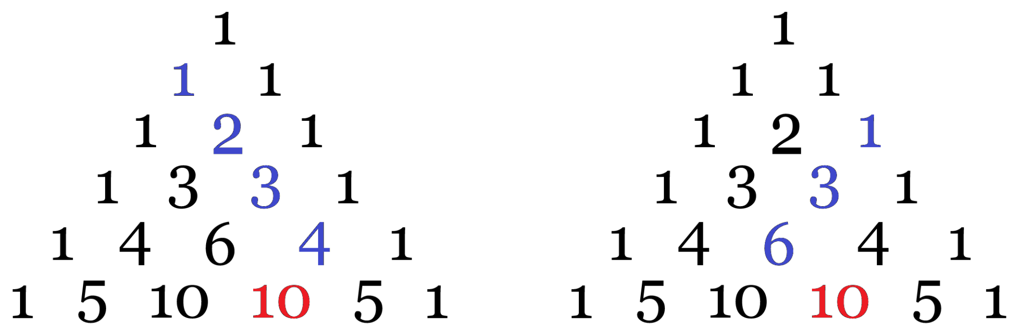
Hockey-Stick Identity. The *hockey-stick identity* states that

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{r}{r} + \binom{r+1}{r} + \ldots + \binom{n-1}{r} + \binom{n}{r} = \binom{n+1}{r+1}.$$

This can also be stated as

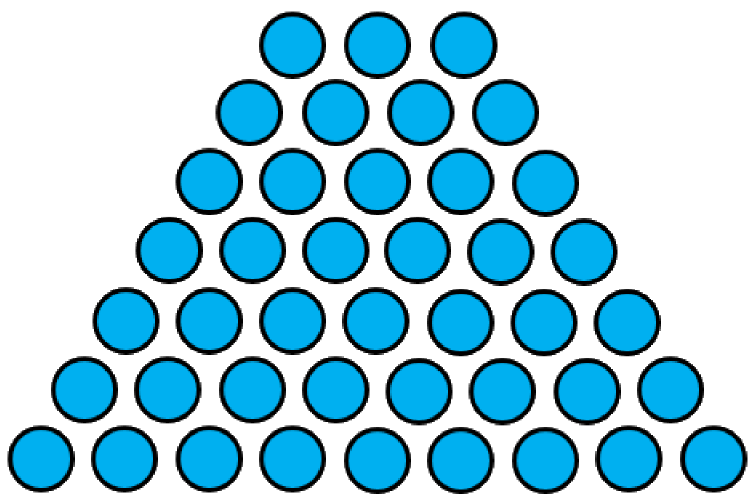
$$\sum_{i=0}^{k} \binom{n-i}{k-i} = \binom{n}{k} + \binom{n-1}{k-1} + \ldots + \binom{n-k+1}{1} + \binom{n-k}{0} = \binom{n+1}{k}.$$

The identity gets its name from its graphical representation on Pascal's triangle:



where the sum of the numbers in blue equals the number in red. The shape of the colored numbers vaguely resembles a hockey-stick.

Example 2.3. Treating each of the balls in the figure below as distinct, how many ways are there to select 3 balls from the same horizontal row?



Solution. The smallest row has 3 balls and the largest row has 9 balls. The number of ways to select 3 balls from the same row can be expressed as a sum of binomial coefficients. This can then be computed with the hockey stick identity:

$$\sum_{k=3}^{9} \binom{k}{3} = \binom{10}{4} = 210$$

So, there are $\boxed{210}$ ways to select 3 balls from the same row.

Binomial Theorem. For any integer n with $n > 0$, we can expand a power of $x + y$ into a sum of the form

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.\end{aligned}$$

Example 2.4. How do you find the term in x^5 in the expansion of $\left(\frac{1}{x} + x\right)^9$?

Solution. By the binomial theorem,

$$(y+x)^9 = \sum_{k=0}^{9}\binom{9}{k}y^k x^{9-k}$$

For $y = \frac{1}{x} = x^{-1}$, this becomes

$$\sum_{k=0}^{9}\binom{9}{k}x^{-k}x^{9-k} = \sum_{k=0}^{9}\binom{9}{k}x^{9-2k}$$

The only way to get x^5 is for $9 - 2k = 5$, that is, $k = 2$, so the coefficient is $\binom{9}{2} = \boxed{36}$.