Probability Solutions

Math Circle Competition Team

September 24th, 2017

$*1.$ Find n if $\binom{10}{4} + \binom{10}{3} = \binom{n}{4}$.

Solution.
$\boxed{n = 11}$ We know by recursion that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Here, $k = 4$ and $n - 1 = 10$, so $\boxed{n = 11}$.

$*2.$ **(1988 AHSME)** For any real number a and any positive integer k, define

$$\binom{a}{k} = \frac{a(a-1)(a-2)\cdots(a-(k-1))}{k!}.$$

What is the exact numerical value of $\frac{\binom{-1/2}{100}}{\binom{1/2}{100}}$?

Solution.
$\boxed{-199}$ We have that

$$\begin{aligned}\frac{\binom{-1/2}{100}}{\binom{1/2}{100}} &= \frac{\frac{(-1/2)(-3/2)\cdot\ldots\cdot(-197/2)(-199/2)}{100!}}{\frac{(1/2)(-1/2)\ldots(-197/2)}{100!}} \\ &= \frac{-199/2}{1/2} \\ &= \boxed{-199}.\end{aligned}$$

$*3.$ Find the number of zeroes at the end of $\binom{200}{100}$.

Solution.
$\boxed{1}$ We know that $\binom{200}{100} = \frac{(200)(199)\cdot\ldots\cdot(101)}{(100)(99)\cdot\ldots\cdot(1)}$. Since there are as many factors of 2 and 5 between 1 and 100 as there are between 101 and 200, match up 200 with 100, 199 with 99, etc. Then, for example, if 195 has a factor of 5, so will 95, so that the factor of 5 cancels, and similarly for pairs which have factors of 2. This is true for every pair except for two: 200 and 100, and 125 and 25. 200/100 leaves a 2, while 125/25 leaves a 5, so that there is exactly one factor of 10 in $\binom{200}{100}$. Thus there is $\boxed{1}$ zero at the end of its expansion.

$*4.$ **(2004 HMMT Feb. Guts)** Find the value of

$$\binom{6}{1}2^1 + \binom{6}{2}2^2 + \binom{6}{3}2^3 + \binom{6}{4}2^4 + \binom{6}{5}2^5 + \binom{6}{6}2^6.$$

Solution.
$\boxed{3^6 - 1 \text{ or } 728}$ From the Binomial Theorem, we know that $\binom{6}{0}2^0 + \binom{6}{1}2^1 + \binom{6}{2}2^2 + \binom{6}{3}2^3 + \binom{6}{4}2^4 + \binom{6}{5}2^5 + \binom{6}{6}2^6 = \sum_{k=1}^{6} \binom{6}{k} 2^k 1^{6-k} = (2+1)^6 = 3^6$. But we must substract the $\binom{6}{0}2^0 = 1$ term, to receive $\boxed{3^6 - 1}$.

$*5.$ Find the coefficient of x^4y in the expansion of $(2x - 3y)^5$.

Solution.
$\boxed{-240}$ Use the Binomial Theorem.

6. What is the coefficient of x^2 in the expansion of $\left(x + \frac{1}{x^2}\right)^{20}$?

Solution.
$\boxed{\binom{20}{14} = 38760}$ Factor out $\frac{1}{x^{40}}$ and use the Binomial Theorem.

7. Pat writes all the 7-digit numbers in which all the digits are different and each digit is greater than the one to its right (so the tens digit is greater than the units, the hundreds greater than the tens, and so on). For example, 9,865,320 is one of the numbers that Pat writes down.

   (a) How many numbers does Pat write down?

   Solution.
   Among any 7 different digits, exactly one number can be formed in which the digits are strictly decreasing (as in 9764321). Therefore, counting the number of these 7-digit all-decreasing numbers is the same as counting the number of ways to select 7 different digits. Since there are 10 digits total, there are $\binom{10}{7} = \boxed{120}$ ways to form a seven digit number such that its digits are strictly decreasing.

   (b) One of Pat's numbers is chosen at random. What is the probability that the tens digit is a 1?

   Solution.
   The only way the tens digit could be a 1 is if both 1 and 0 are among the digits chosen. Then the 0 will be in the units and the 1 in the tens. There are $\binom{8}{5} = 56$

7-digit numbers of the form _____10, because there are 56 ways to choose 5 of the remaining 8 digits to form the number. Therefore, the desired probability is $\frac{56}{120} = \boxed{\dfrac{7}{15}}$.

(c) One of Pat's numbers is chosen at random. What is the probability that the middle (thousands) digit is a 5?

Solution.
We follow the logic hinted at by part (b). If 5 is in the middle, there are $\binom{5}{3} = 10$ ways to choose the last three digits (from 0,1,2,3,4) and $\binom{4}{3} = 4$ ways to choose the first three digits (from 6,7,8,9), for a total of $10 \times 4 = 40$ different ways 5 can be in the middle. Therefore, the desired probability is $\frac{40}{120} = \boxed{\dfrac{1}{3}}$.

*8. **(2016 HMMT Nov. General)** I have five different pairs of socks. Every day for five days, I pick two socks at random without replacement to wear for the day. Find the probability that I wear matching socks on both the third day and the fifth day.

Solution.
$\boxed{1/63}$ I get a matching pair on the third day with probability 1/9 because there is a 1/9 probability of the second sock matching the first. Given that I already removed a matching pair on the third day, I get a matching pair on the fifth day with probability 1/7. We multiply these probabilities to get $\boxed{1/63}$.

*9. **(2009 AMC 10A)** Three distinct vertices of a cube are chosen at random. What is the probability that the plane determined by these three vertices contains points inside the cube?

Solution.
$\boxed{4/7}$ There are $\binom{8}{3} = 56$ ways to pick three vertices from eight total vertices; this is our denominator. In order to have three points inside the cube, they cannot be on the surface. Thus, we can use complementary probability. There are four ways to choose three points from the vertices of a single face. Since there are six faces, $4 \times 6 = 24$. Thus, the probability of what we don't want is $\frac{24}{56} = \frac{3}{7}$. Using complementary probability, the answer is then $1 - \dfrac{3}{7} = \boxed{\dfrac{4}{7}}$.

10. **(2010 AMC 10A)** Bernardo randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

Solution.
$\boxed{37/56}$ We can solve this by breaking the problem down into 2 cases and adding up the probabilities.

(a) *Case* 1: Bernardo picks 9. If Bernardo picks a 9 then it is guaranteed that his number will be larger than Silvia's. The probability that he will pick a 9 is $\frac{1 \cdot \binom{8}{2}}{\binom{9}{3}} = \frac{1}{3}$.

(b) *Case* 2: Bernardo does not pick 9. Since the chance of Bernardo picking 9 is $\frac{1}{3}$, the probability of not picking 9 is $\frac{2}{3}$.

If Bernardo does not pick 9, then he can pick any number from 1 to 8. Since Bernardo is picking from the same set of numbers as Silvia, the probability that Bernardo's number is larger is equal to the probability that Silvia's number is larger. Ignoring the 9 for now, the probability that they will pick the same number is the number of ways to pick Bernardo's 3 numbers divided by the number of ways to pick any 3 numbers. We get this probability to be $\frac{3!}{8 \cdot 7 \cdot 6} = \frac{1}{56}$. The probability of Bernardo's number being greater is

$$\frac{1 - \frac{1}{56}}{2} = \frac{55}{112}.$$

Factoring in the fact that Bernardo could've picked a 9 but didn't, we receive

$$\frac{2}{3} \cdot \frac{55}{112} = \frac{55}{168}.$$

Adding up the two cases, our final answer is $\frac{1}{3} + \frac{55}{168} = \boxed{\frac{37}{56}}$

11. **(2010 AIME II)** Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After waiting at that gate, Dave is told the departure gate has been changed to a different gate, again at random. Let the probability that Dave walks 400 feet or less to the new gate be a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

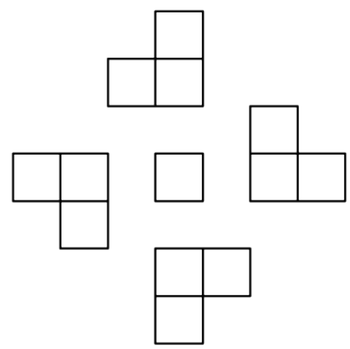
Solution.
$\boxed{52}$ There are $12 \cdot 11 = 132$ possible situations (12 choices for the initially assigned gate, and 11 choices for which gate Dave's flight was changed to). We are to count the situations in which the two gates are at most 400 feet apart.
If we number the gates 1 through 12, then gates 1 and 12 have four other gates within 400 feet, gates 2 and 11 have five, gates 3 and 10 have six, gates 4 and 9 have have seven, and gates 5, 6, 7, 8 have eight. Therefore, the number of valid gate assignments

is

$$2 \cdot (4+5+6+7) + 4 \cdot 8 = 2 \cdot 22 + 4 \cdot 8 = 76$$

so the probability is $\dfrac{76}{132} = \dfrac{19}{33}$. The answer is $19 + 33 = \boxed{52}$.

12. **(2013 AIME I)** In the array of 13 squares shown below, 8 squares are colored red, and the remaining 5 squares are colored blue. If one of all possible such colorings is chosen at random, the probability that the chosen colored array appears the same when rotated 90 degrees around the central square is $\frac{1}{n}$, where n is a positive integer. Find n.



Solution.
$\boxed{429}$ When the array appears the same after a 90 degree rotation, the top formation must look the same as the right formation, which looks the same as the bottom one, which looks the same as the right one. There are four of the same configuration. There are not enough red squares for these to be all red, nor are there enough blue squares for there to be more than one blue square in each three-square formation. Thus there are 2 reds and 1 blue in each, and a blue in the center. There are 3 ways to choose which of the squares in the formation will be blue, leaving the other two red. There are $\binom{13}{5}$ ways to have 5 blue squares in an array of 13, so we get

$$\frac{3}{\binom{13}{5}} = \frac{1}{429}.$$

Thus we have $n = \boxed{429}$.

13. A playoff series between two teams proceeds one game at a time until one team has won 5 games. What is the probability that the series lasts 9 games if each team is equally likely to win each game?

Solution.
$\boxed{35/128}$ The last game must be a win, as this will end the series. Thus we have 8 games and must choose 4 as wins. The total number of possible arrangements is 2^8

(since each game can be either a win or a loss), and the number of arrangements with exactly 4 wins is $\binom{8}{4}$. Thus our final answer is $\frac{\binom{8}{4}}{2^8} = \boxed{\frac{35}{128}}$.

14. **(2016 HMMT Nov. General)** The numbers $1, 2, \ldots, 11$ are arranged in a line from left to right in a random order. It is observed that the middle number is larger than exactly one number to its left. Find the probability that it is larger than exactly one number to its right.

Solution.
$\boxed{10/33}$ Suppose the middle number is k. Then there are $k-1$ ways to pick the number smaller than k to its left and $\binom{11-k}{4}$ ways to pick the 4 numbers larger than k to its right. Hence there is a total of $\sum_{k=2}^{7}(k-1) \cdot \binom{11-k}{4}$ ways for there to be exactly one number smaller than k to its left. We calculate this total:

$$\begin{aligned}\sum_{k=2}^{7}(k-1) \cdot \binom{11-k}{4} &= \sum_{j=4}^{9}\sum_{i=4}^{j}\binom{i}{4} \\ &= \sum_{j=4}^{9}\binom{j+1}{5} \\ &= \binom{11}{6}.\end{aligned}$$

The only way k can be larger than exactly one number to its right is if $k = 3$. Then the probability of this happening is $\frac{2 \cdot \binom{8}{4}}{\binom{11}{6}} = \boxed{\frac{10}{33}}$.