Geometry Day 1 Notes

Math Circle Competition Team
October 8th, 2017

Metric Relations and Triangle Trivia

The Triangle: In Euclidean space, we have the notion of a **triangle**, defined by three vertices connected by line segments. Most of Olympiad geometry is concerned with triangles, and they have many interesting properties.
We will use the following notation for the side lengths of $\triangle ABC$: $BC = a$, $CA = b$, and $AB = c$. It is generally helpful to use notation is that symmetric, as we will see in our introduction of the triangle centers.

Circumcenter: The center of the circle which passes through all three vertices of a triangle. It is also the intersection of the three perpendicular bisectors of a triangle's sides.

Proof: Suppose ℓ_a and ℓ_b are the perpendicular bisectors of $\overline{BC}$ and $\overline{CA}$ respectively, and let these two lines intersect at O. We wish to show that O also lies on the perpendicular bisector of $\overline{AB}$. Note that since O lies on l_a, $OB = OC$. Similarly, since O lies on l_b we have $OC = OA$. Therefore $OA = OB$, meaning that O lies on the perpendicular bisector of $\overline{AB}$ as desired.

- Note that this overlooks a technical point that the two lines ℓ_a and ℓ_b might not intersect. To prove that this can never happen, we use proof by contradiction. Suppose that it did occur. Then since $\ell_a \perp BC$ and $\ell_b \perp AC$, we must have $BC \parallel AC$, meaning that A, B, and C are collinear. This is a contradiction.

Incenter: The center of the circle which is tangent to each of three sides of a triangle. It is also the intersection of the three angle bisectors of a triangle.

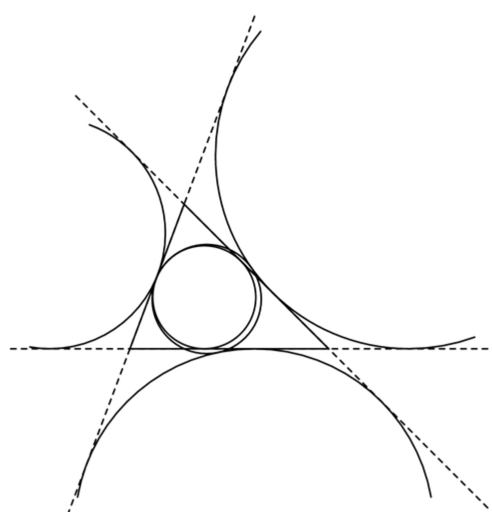
Proof: Suppose the angle bisectors of $\angle A$ and $\angle B$ intersect at a point I; then it suffices to prove that I passes through the angle bisector of $\angle C$. Suppose the foot of the perpendicular from I to BC is A'; define B' and C' similarly. Remark that since I lies on the angle bisector from $\angle A$ we have $IB' = IC'$. Similarly, since I lies on the angle bisector from $\angle B$ we have $IA' = IC'$. Therefore by transitivity $IA' = IB'$, so I lies on the angle bisector from $\angle C$ as desired.

- Note that this overlooks a technical point that the two angle bisectors might not intersect. To prove this can never happen, we use proof by contradiction. Suppose that it did occur. Then $\angle IAB$ and $\angle IBA$ are corresponding angles with regard to the transversal AB, and so they must be supplementary to each other. But this implies $\angle AIB = 180^\circ - \angle IAB - \angle IBA = 0^\circ$. This is a contradiction.

Are there any other circles that are tangent to the three sides of any triangle? There are actually three other such circles!

Excircles: The excircles of the triangle are tangent externally to three sides of a triangle. The **excenters** are defined in a similar manner as the incenter; for example, the A-excenter, I_A, is defined as the intersection point of the internal bisector of $\angle A$ and the external bisectors of $\angle B$ and $\angle C$.

Side Note: Another circle which is important to the triangle is the **nine point circle**. Given a triangle ABC, there exists a circle which passes through the midpoints of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$ (where H is the orthocenter). This configuration has many properties - for example, it turns out that the incircle and three excircles are all tangent to the nine-point circle! This will not be proven now but can be seen in the picture below:



Within a triangle there are several metric relations that relate various scalars such as side lengths, angle measures, and area. Understanding these relationships will help solve normally difficult geometry problems.

The Pythagorean Theorem: For a right triangle ABC with legs of length a and b and hypotenuse of length c we have $a^2 + b^2 = c^2$.

- Try to prove this on your own! The Pythagorean theorem is a classic theorem, and there exist hundreds of proofs for it using all sorts of configurations that range from elementary to obscure.

The Law of Sines: In triangle ABC,

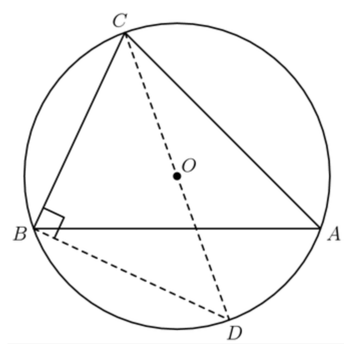
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Proof: Construct the altitude from B to side AC and denote it h. It can be seen from the definition of sine that $\sin A = \frac{h}{c}$ and $\sin C = \frac{h}{a}$. Thus, $h = c \sin A$ and $h = a \sin C$. This gives $c \sin A = a \sin C$ and $\frac{a}{\sin A} = \frac{c}{\sin C}$. We can repeat this cyclically for altitudes from A and C.

The (Extended) Law of Sines: In $\triangle ABC$, where R is the circumradius:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Proof: In the diagram below, point O is the circumcenter of $\triangle ABC$. Point D is on BC such that $OD \perp BC$. Since $\triangle ODB \cong \triangle ODC$, $BD = CD = \frac{a}{2}$ and $\angle BOD = \angle COD$. But $2\angle BAC = \angle BOC$ making $\angle BOD = \angle COD = \theta$. We can use simple trigonometry in right triangle $\triangle BOD$ to find that $\sin \theta = \frac{\frac{a}{2}}{R} \iff \frac{a}{\sin \theta} = 2R$. The same holds for b and c, thus establishing the identity.



The Law of Cosines: In triangle ABC,

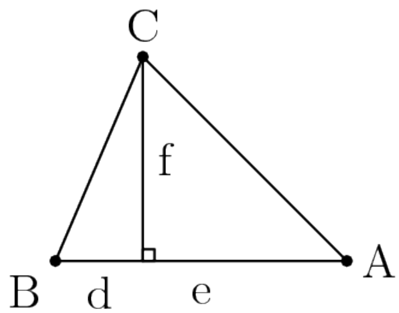
$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Note that in the case that one of the angles has measure $90°$ ($\triangle ABC$ is a right triangle), the corresponding statement reduces to the Pythagorean Theorem.

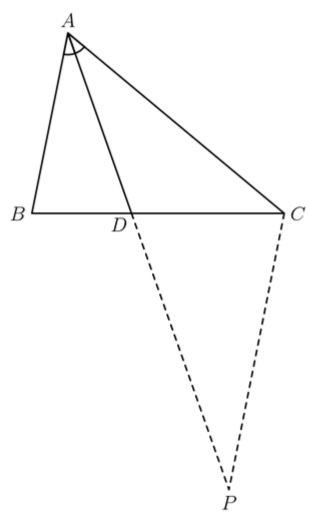
- Try to prove this on your own! Observe the diagram below. If you drop an altitude from any vertex, say C, note that it will divide the opposite side AB into two different lengths d and e. Express the sum of the squares of these lengths in terms of a, b, and the length of the altitude that you just created. Now finish with the definition of cosine on $\angle C$.



The Angle Bisector Theorem: The angle bisector of $\angle A$ of the triangle ABC intersects the opposite side BC at D. We have

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

Proof: Let the internal bisector of $\angle A$ intersect BC at D and the line through C parallel to AB at P, as shown.



Since $AB \parallel CP$ we have $\triangle ABD \sim \triangle DCP$, so $\frac{AB}{AD} = \frac{CP}{CD}$. However, take further note that $\angle CAD = \angle BAD = \angle CPD$, so $\triangle ACP$ is isosceles and $AC = CP$. Hence $\frac{AB}{AD} = \frac{AC}{CD}$; rearranging gives our desired result.

Ratio Lemma: In $\triangle ABC$, let D be a point on BC. If $\angle BAD = \alpha, \angle CAD = \beta$ then

$$BD/CD = AB \sin\alpha / AC \sin\beta.$$

Proof: By the Law of Sines on $\triangle ACD$ we have

$$\frac{\sin\beta}{CD} = \frac{\sin\angle CDA}{AC},$$

and by LoS on $\triangle ABD$ gives

$$\frac{\sin\alpha}{BD} = \frac{\sin(180 - \angle CDA)}{AB} = \frac{\sin\angle CDA}{AB}.$$

Dividing the first equation by the second and rearranging gives the desired result.

Useful Area Formulas: Denote the area of a triangle ABC by $[ABC]$. Then we have the following formulas:

1. $[ABC] = \frac{1}{2}bc\sin\angle A$.
2. $[ABC] = rs$ where r is the inradius and $s = \frac{a+b+c}{2}$ is the semiperimeter of $\triangle ABC$.
3. $[ABC] = \frac{abc}{4R}$ where R is the circumradius.
4. **Pick's Theorem:** Suppose that a polygon with integer vertices contains I integer points in its interior and B integer points along its edges. These are also known as lattice points. Then the area K of the polygon is $K = I + \frac{B}{2} - 1$.

Proofs:

1. Let the foot of the altitude from C intersect AB at a point D. We know that $[ABC] = \frac{1}{2} \cdot CD \cdot AB = \frac{1}{2}c \cdot CD$. However, we can also write $CD = b\sin A$ from right triangle trigonometry. Hence $[ABC] = \frac{1}{2}bc\sin A$ as desired.
2. Suppose the incircle of $\triangle ABC$ is tangent to BC, CA, and AB at A_1, B_1, and C_1 respectively, and let the incenter of $\triangle ABC$ be I. Then by breaking up the area of the triangle we obtain

$$\begin{aligned}[ABC] &= [BIC] + [CIA] + [AIB] \\ &= \frac{1}{2} \cdot IA_1 \cdot BC + \frac{1}{2} \cdot IB_1 \cdot CA + \frac{1}{2} \cdot IC_1 \cdot AB \\ &= \frac{1}{2}r(a+b+c) = rs.\end{aligned}$$

3. It suffices to recall the extended Law of Sines, plug $\sin\angle A = \frac{a}{2R}$ into the area formula $[ABC] = \frac{1}{2}bc\sin\angle A$ and rearrange.
4. If a triangle is on three lattice points with no point in its interior or on its edges, it has an area of $\frac{1}{2}$. Such a triangle must contain two lattice points distance 1 from each other and one lattice point on a line parallel to the opposite edge distance 1 apart. The minimum distance between two distinct lattice points is 1. If no two lattice points have distance 1, by $\frac{1}{2}bh$ the area is more than 1 and similarly for the height. Removing 1 of the mentioned triangles either removes 1 boundary point or turns 1 interior point into a boundary point, accounting for the $I + \frac{1}{2}B$ part. The -1 part is accounted for by looking at the area of the unit triangle with 3 boundary points, 0 interior points, and $\frac{1}{2}$ area.

Exradii and Area: We proved that $[ABC] = rs$. Prove an analogous formula concerning the circle that is tangent to BC, the extensions of AB beyond B, and the extension of AC beyond C. This circle is called the A-excircle of $\triangle ABC$ and its radius is denoted by r_a.

Proof: Let I_a be the center of the A-excircle. Then

$$[ABC] = [ABI_a] + [ACI_a] - [BCI_a] = \frac{1}{2}c \cdot r_a + \frac{1}{2}b \cdot r_a - \frac{1}{2}a \cdot r_a = r_a(s-a).$$

Equal Tangents: Point A lies outside of circle ω. Lines AT, AU are tangent to the circle at T, U, respectively. Then $AT = AU$.

Proof: Let O be the center of ω. Then $OT = OU$ and $\angle ATO = \angle AUO = 90°$, so $\triangle ATO = \triangle AUO$ and the conclusion follows.

Properties of Incircle and Excircle:

1. Let the incircle of $\triangle ABC$ touch BC, CA, AB at D, E, F, respectively. Then

$$AE = AF = s - a, \qquad BF = BD = s - b, \quad CD = CE = s - c.$$

   Note: We usually denote these distances by $s - a = x$, $s - b = y$, $s - c = z$.

   Proof: Note that by Equal Tangents we have $AE = AF = x$, $BF = BD = y$, and $CD = CE = z$. Note that as a result we have $a = y+z$, $b = z+x$, $c = z+y$. Now remark that $2(x+y+z) = a+b+c = 2s \implies x+y+z = s$. Thus $x = s-(y+z) = s-a$. Similar reasoning on the other two equalities gives the desired.

2. Let the A-excircle of $\triangle ABC$ touch BC, CA, AB at D_1, E_1, F_1, respectively. Then

$$AE_1 = AF_1 = s, \qquad BF_1 = BD_1 = s - c, \qquad CD_1 = CE_1 = s - b.$$

   Proof: Let $BA_1 = BF_1 = x$, $CA_1 = CE_1 = y$, and $AF_1 = AE_1 = z$. Remark that $a = x + y$, $b = z - x$, and $c = z - y$. Thus

$$a + b = 2z - (x + y) = 2z - c \implies 2z = a + b + c = 2s \implies s = z.$$

   The conclusion now follows from substitution.

3. Points D, D_1 are symmetric about the midpoint M of the side BC.

   Proof: From the previous two exercises we have $BD = CD_1 = s - b$. Thus D and D_1 are equidistant from the respective endpoints of the segment $[BC]$, meaning that the two points are symmetric about the midpoint of $[BC]$ as desired.

4. $[ABC] = r_a \cdot (s - a)$ where r_a is the radius of the A-excircle of $\triangle ABC$. We've shown that the proof of this mimics the proof of the formula $[ABC] = rs$.

Stewart's Theorem: Let ABC be a triangle and D a point on its side BC with $BD = m$ and $CD = n$. Then

$$dad + man = bmb + cnc.$$

Proof: We use Law of Cosines on $\triangle ABD$ and $\triangle ACD$. Let $\theta = \angle ADB$. Using the two angles with vertex D as our angles of reference, we obtain the two equations

$$c^2 = d^2 + m^2 - 2dm\cos\theta, \qquad b^2 = d^2 + n^2 + 2dn\cos\theta.$$

Stewart's Theorem is in terms of only the side lengths, though, so we wish to get rid of θ. To do this, we multiply the first equation by n and the second equation by m and add the two equations together; this gets rid of the cosine terms and leaves us with

$$nc^2 + b^2m = d^2m + d^2n + m^2n + mn^2 = d^2(m+n) + mn(m+n) = d^2a + mna.$$

Rearranging individual terms within each monomial gives the desired.

- Note that if if D lies outside $\triangle ABC$, we can take CA as the cevian in $\triangle CBD$. This gives a "new" formula,

  $$CD^2 \cdot AB = CA^2 \cdot DB + DB \cdot DA \cdot AB - CB^2 \cdot AD.$$

- We can combine both of the above cases using directed segments. Directed segments are similar to the notion of length except for direction; in other words, $\overline{AB} = -\overline{BA}$. This formula happens to be

  $$CD^2 \cdot \overline{AB} + CA^2 \cdot \overline{BD} + CB^2 \cdot \overline{DA} + \overline{DA} \cdot \overline{AB} \cdot \overline{BD} = 0.$$

  To prove this, simply check a few cases (two of them are mentioned above) to see if the formulas match. There is a more direct way to prove this using vectors, but we won't be covering that now.

Heron's Formula: In triangle ABC,

$$[ABC]^2 = s(s-a)(s-b)(s-c)$$

where s is again the semiperimeter of $\triangle ABC$.

Proof: Suppose WLOG that $\alpha \leq \beta \leq \gamma$. This ensures that the foot of the altitude from C lies on segment $\overline{AB}$. Let $BH_c = x$, where X is said foot. Then by Pythagorean Theorem

$$a^2 - x^2 = b^2 - (c-x)^2 \implies x = \frac{a^2 - b^2 + c^2}{2c}.$$

Thus $CH_c^2 = a^2 - (\frac{a^2-b^2+c^2}{2c})^2$. It remains an exercise to the reader to perform the necessary computations to show that $\frac{1}{2}c \cdot h_c$ is precisely $\sqrt{s(s-a)(s-b)(s-c)}$.

Dual Principle: If a, b, c are the sides of a triangle ABC, then we can write $a = y + z$, $b = z + x$, and $c = x + y$ where $x, y, z > 0$.

Proof: Take $x = s - a$, $y = s - b$, $z = s - c$ and observe they are positive by the triangle inequality.

Worked Examples:

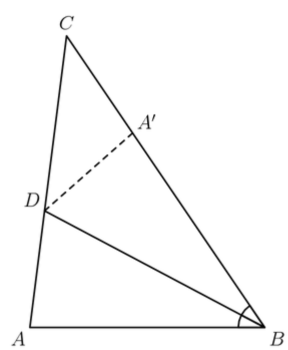
1. In triangle ABC, $AB = 12$, $BC = 18$ and $AC = 15$. Prove that $\angle A = 2 \cdot \angle C$.

   Solution 1: Suppose the angle bisector of $\angle A$ intersects BC at point D. Then by the Angle Bisector Theorem we obtain $CD = 10$, $DB = 8$. In order to prove that $\angle A = 2\angle C$ it suffices to prove that $\triangle ADC$ is isosceles; this way, $\angle C = \angle CAD = \frac{1}{2}\angle A$ and the result follows. We do this using Stewart's Theorem:

$$\begin{aligned}15^2 \cdot 8 + 12^2 \cdot 10 &= AD^2 \cdot 18 + 18 \cdot 10 \cdot 8\\ 1800 + 1440 &= 18AD^2 + 1440\\ 18AD^2 &= 1800\\ BD &= 10.\end{aligned}$$

   This proves the original statement.

   Solution 2: While the solution above is quick, it is not very elegant. Here is a solution that explores deeper into the configuration. Let the angle bisector of $\angle CBA$ intersect AC at E, and let A' denote the reflection of A over BD, as shown:



   We first claim that $AB = A'B$ and $AE = A'E$. Let $AA' \cap BD = X$. Then from the definition of reflection $AA' \perp BX$, so X is both the altitude and the angle bisector from B in $\triangle ABA'$. Thus $\triangle BAX = \triangle BA'X$, and so $BA = BA'$. A similar argument yields $AE = A'E$. Scale down the original problem so that $AB = 4$, $BC = 6$, and $AC = 5$. Now from the angle bisector theorem conclude that $AD = 2$ and $CD = 3$. This means that $AD = 2$ and $A'C = BC - BA' = 6 - 4 = 2$. As a result, $\triangle CA'D$ is isosceles. Now

$$\angle A = \angle DA'B = \angle A'CD + \angle A'DC = 2\angle C,$$

which is what we wanted.

2. Point D lies on base BC of equilateral triangle ABC. Prove that the circumradii of triangles ABD and ACD are equal. Generalize to some other class of triangles.

   Solution: Denote the circumradii of $\triangle ABD$, $\triangle ACD$ by R_b, R_c, respectively. Since $\angle ABD = \angle ACD = 60°$, employing the extended Law of Sines twice we have

$$2R_b = \frac{AD}{\sin\angle ABD} = \frac{AD}{\sin\angle ACD} = 2R_c,$$

   and the result follows.

   The same argument works if $\triangle ABC$ is A-isosceles.

3. Prove that the area of a convex quadrilateral $ABCD$ with angle ϕ between the diagonals can be computed as

$$[ABCD] = \frac{1}{2}\cdot AC\cdot BD\cdot\sin\phi.$$

   Does this still hold true when the quadrilateral is nonconvex?

   Solution 1: Denote the intersection of diagonals by P and recall that $\sin\phi = \sin(180° - \phi)$. Now it suffices to split the quadrilateral into four triangles and apply the area formula for each of them:

$$\begin{aligned}[ABCD] &= [ABP] + [BCP] + [CDP] + [DAP] = \\ &= \frac{1}{2}(PA\cdot PB + PB\cdot PC + PC\cdot PD + PD\cdot PA)\sin\phi = \\ &= \frac{1}{2}\sin\phi(PA+PC)(PB+PD) = \frac{1}{2}AC\cdot BD\cdot\sin\phi.\end{aligned}$$

   Solution 2: Let A' be the point such that $ADA'C$ is a parallelogram. Then one can prove that $[BDA'] = [ABCD]$, and the area of the latter is $\frac{1}{2}BD\cdot DA'\sin\phi = \frac{1}{2}BD\cdot AC\sin\phi$. (The proof of this fact is left to the reader!)

   For the nonconvex case, we see that that formula still holds true. To see this, we utilize another proof given in class. Let $ABCD$ be a quadrilateral with $\angle BCD > 180°$, and let $X = AC\cap BD$. Furthermore, let h_b and h_d be the lengths of the perpendiculars from B and D to AC. Then

$$\begin{aligned}[ABCD] = [ABC] + [ACD] &= \frac{1}{2}AC\cdot h_b + \frac{1}{2}AC\cdot h_d) \\ &= \frac{1}{2}AC(DX\sin\phi + BX\sin\phi) = \frac{1}{2}AC\cdot BD\sin\phi.\end{aligned}$$

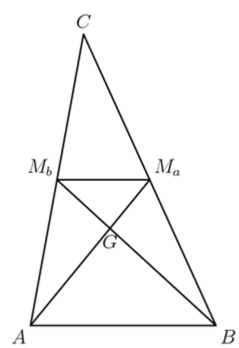
4. Let ABC be a triangle and M be the midpoint of BC. Then m_a is the standard notation for the length AM. Prove that $m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2)$.

   Solution: Plugging $d = m_a$ and $m = n = \frac{1}{2}a$ into the Stewart's Theorem gives

   $$a \cdot m_a^2 + \frac{1}{4}a^3 = \frac{a}{2}b^2 + \frac{a}{2}c^2.$$

   Cancel a and rearrange.

5. Prove that medians AA_1 and BB_1 of triangle ABC are perpendicular if and only if $a^2 + b^2 = 5c^2$.

   Solution 1: Let G be the centroid of $\triangle ABC$. Since $AG = \frac{2}{3}AA_1$ and $BG = \frac{2}{3}BB_1$, the triangle ABG is right if and only if

   $$\left(\frac{2}{3}m_a\right)^2 + \left(\frac{2}{3}m_b\right)^2 = c^2$$

   which, using the median formulas, rewrites as

   $$(2b^2 + 2c^2 - a^2) + (2a^2 - b^2 + 2c^2) = 9c^2.$$

   Solution 2: Consider the median from C instead. Use $a^2 + b^2 = 5c^2$ and the median formula to compute

   $$m_c^2 = \frac{1}{4}(2a^2 + 2b^2 - c^2) = \frac{9}{4}c^2 \implies m_c = \frac{3}{2}c.$$

   Thus, if G is the centroid of $\triangle ABC$ and M the midpoint of $\overline{BC}$, we obtain $MG = \frac{1}{2}c = MA = MB$. This implies the conclusion.

6. Prove that $[ABC]^2 = rr_ar_br_c$.

   Solution 3: Suppose medians AM_a and BM_b intersect at the centroid G of $\triangle ABC$.

Note that quadrilateral ABM_aM_b is orthodiagonal[1], so we may use the formula mentioned above. Remark that $M_aM_b = \frac{1}{2}c$, $AM_b = \frac{1}{2}b$, and $BM_a = \frac{1}{2}c$. Combining this with $AB = c$ we obtain

$$\left(\frac{1}{2}c\right)^2 + c^2 = \left(\frac{1}{2}a\right)^2 + \left(\frac{1}{2}b\right)^2 \iff 5c^2 = a^2 + b^2.$$

Solution: Denote the area of $\triangle ABC$ by K. Using the first part of this problem and the Heron's formula we may write

$$rr_ar_br_c = \frac{K}{s} \cdot \frac{K}{s-a} \cdot \frac{K}{s-b} \cdot \frac{K}{s-c} = \frac{K^4}{K^2} = K^2.$$

7. Through point P, lines a, b, c, d are drawn. A line l intersects these lines at A, B, C, D, respectively. Prove that the quantity

$$\frac{AC}{CB} : \frac{AD}{DB}$$

does not depend on the choice of line l. This quantity is called the cross-ratio and is denoted by $(AB;CD)$.

Solution: Since all the triangles ACP, BDP, BCP, ADP share the altitude we may rewrite

$$\frac{AC \cdot BD}{BC \cdot AD} = \frac{[ACP] \cdot [BDP]}{[BCP] \cdot [ADP]}.$$

Denoting $\angle APB = \phi$, $\angle BPC = \psi$, $\angle CPD = \theta$ this may be further rewritten as

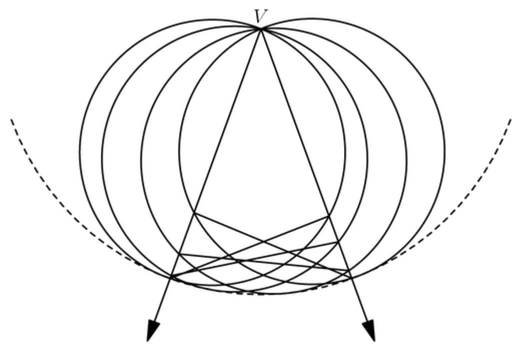
$$\frac{\left(\frac{1}{2}PA \cdot PC \cdot \sin(\phi+\psi)\right) \cdot \left(\frac{1}{2}PB \cdot PD \cdot \sin(\psi+\theta)\right)}{\left(\frac{1}{2}PB \cdot PC \cdot \sin(\psi)\right) \cdot \left(\frac{1}{2}PA \cdot PD \cdot \sin(\phi+\psi+\theta)\right)} = \frac{\sin(\phi+\psi) \cdot \sin(\psi+\theta)}{\sin(\psi) \cdot \sin(\phi+\psi+\theta)}.$$

Hence the quantity does not depend on the position of line l and we may conclude.

Note: Instead of transforming the ratios to the ratios of areas we could have used Law of Sines.

8. Given an acute angle with vertex V and a positive real number d, consider all the segments XY of lenght d whose endpoints lie on different sides of the angle. Prove that the circumcircles of all the triangles XYV are tangent to some fixed circle.

[1] i.e. the diagonals are perpendicular



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Solution: By Law of Sines in $\triangle XVY$, the circumradius R of the triangle XVY is constant (indeed, it satisfies $2R = \frac{XY}{\sin \angle XVY}$). Hence all the circles are internally tangent to the circle ω centered at V and with (fixed) radius $2R$.

9. Prove $r_a + r_b + r_c = 4R + r$, where r_a denotes the A-exradius.

   Solution: Taking motivation from the Dual Principle and using $K = [ABC]$, we may rewrite
$$\frac{K}{x} + \frac{K}{y} + \frac{K}{z} = \frac{(x+y)(y+z)(z+x)}{K} + \frac{K}{s}.$$
   After multiplying by $xyz(x+y+z) = K^2$ and cancelling K this turns into
$$(x+y+z)(xy+yz+zx) = (x+y)(y+z)(z+x) + xyz$$

10. Prove that
$$a^2 + b^2 + c^2 \geq 4\sqrt{3}[ABC] + (b-c)^2 + (c-a)^2 + (a-b)^2.$$

    Solution: Rewriting in terms of x, y, z ($x, y, z > 0$) and squaring we obtain an equivalent inequality
$$(y+z)^2 + (z+x)^2 + (x+y)^2 \geq 4\sqrt{3}\sqrt{xyz(x+y+z)} + (z-y)^2 + (x-z)^2 + (y-x)^2$$
$$(xy+yz+zx)^2 \geq 3xyz(x+y+z)$$

    Substituting $t = xy, u = yz, v = zx$ we recognize the familiar well-known inequality
$$t^2 + u^2 + v^2 \geq tu + uv + vt.$$

11. Prove that $[ABC]^2 = rr_ar_br_c$.

    Solution: Multiplying the area formulas $[ABC] = rs = r_a(s-a) = r_b(s-b) = r_c(s-c)$ and dividing by $s(s-a)(s-b)(s-c) = [ABC]^2$ (Heron's formula) we obtain the result.

12. Lines AT, AU are tangent to the circle ω at T, U, respectively, with $AT = AU = 8$.

Line ℓ tangent to ω intersects the segments AT, AU at X, Y, respectively. Prove that the perimeter of $\triangle AXY$ does not depend on the choice of line ℓ and find it.

Solution: Denote the point of contact of ℓ and ω by D. By Equal Tangents, $(AX + XD) + (DY + YA) = AT + AU = 16$.

13. Suppose that the quadrilateral $ABCD$ is *circumscribed* (i.e. there exists a circle tangent to all its four sides). Prove that $AB + CD = BC + DA$. ℓ and find it.

    Solution: Denoting the points of contact by K, L, M, N, respectively, Equal Tangents imply $AK = AN = x$, $BK = BL = y$, $CL = CM = Z$, $DM = DN = w$. Both sides are then equal to $x + y + z + w$.
    Note: The other implication works too, i.e. if the side lengths of the quadrilateral $ABCD$ satisfy $AB + CD = BC + AD$, then it is circumscribed. For the proof, consider a circle lying inside the quadrilateral tangent to three of its sides and argue by contradiction (use triangle inequality).

14. Prove that
$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$
    Solution: Using the area formulas, the equation rewrites as
$$\frac{s}{[ABC]} = \frac{s-a}{[ABC]} + \frac{s-b}{[ABC]} + \frac{s-c}{[ABC]},$$
    which is true.

15. The incircle of $\triangle ABC$ touches its sides BC, CA, AB at points D, E, F, respectively. Also, the excircles of $\triangle ABC$ touch the corresponding sides of $\triangle ABC$ at points T, U, V. Show that triangles DEF and TUV have the same area.

    Solution: First, we recall $AF = BV = x$, $BD = TC = y$, and $CE = AU = z$. The idea is to express complementary areas in terms of x, y, z and then just check an algebraic equality. Area of each of the small triangles can be expressed using the lengths of two sides and sine of the angle between then. The Law of Sines (in triangle ABC) allows us to rewrite this sine as (for instance) $\sin A = \frac{a}{2R} = \frac{y+z}{2R}$. Hence we obtain
$$\begin{aligned}[AFE] + [BDF] + [CED] &= \frac{1}{2}\left(x^2 \sin\angle A + y^2 \sin\angle B + z^2 \sin\angle C\right) \\ &= \frac{1}{4R}\left(x^2(y+z) + y^2(z+x) + z^2(x+y)\right)\end{aligned}$$

and

$$[AVU]+[BTV]+[CUT]=\frac{1}{2}\left(yz\sin\angle A+zx\sin\angle B+xy\sin\angle C\right)$$
$$=\frac{1}{4R}\left(yz(y+z)+zx(z+x)+xy(x+y)\right).$$

Since the complements are indeed equal, we may conclude.

Another reasoning giving the same result involves observing that

$$\frac{[AFE]}{[ABC]}=\frac{\frac{1}{2}x^2\sin\angle A}{\frac{1}{2}(x+y)(x+z)\sin\angle A}=\frac{x^2}{(x+y)(x+z)},$$

hence the portion of the area of $\triangle ABC$ occupied by triangles AEF, BDF, CED combined equals

$$\frac{x^2(y+z)+y^2(z+x)+z^2(x+y)}{(x+y)(y+z)(z+x)}.$$

Doing the same for TUV we end up checking the same equality as in the first proof.

16. **AIME I 2005 #7** In quadrilateral $ABCD$, $BC=4$, $CD=7$, $AD=1$, and $\angle A=\angle B=60^\circ$. Find the distance AB.

    Solution: Let E be the intersection of lines AD and BC. Then the triangle ABE is equilateral. Hence denoting $AB=x$, we have $CE=x-4$ and $DE=x-1$. As $\cos 60^\circ=\frac{1}{2}$, the Law of Cosines in $\triangle CDE$ yields

    $$7^2=(x-1)^2+(x-4)^2-(x-1)(x-4)=x^2-5x+13,$$
    $$0=(x-9)(x+4).$$

    Thus, $AB=9$.

17. **AIME 1987 #9** Triangle ABC has a right angle at B and contains a point P such that $PA=10$, $PB=6$, and $\angle APB=\angle BPC=\angle CPA$. Find PC.

    Solution: Let $PC=x$. Since $\angle APB=\angle BPC=\angle CPA$, each of them is equal to 120°. By the Law of Cosines applied to triangles $\triangle APB$, $\triangle BPC$ and $\triangle CPA$ at their respective angles P, remembering that $\cos 120^\circ=-\frac{1}{2}$, we have

    $$AB^2=36+100+60=196,\ BC^2=36+x^2+6x,\ CA^2=100+x^2+10x.$$

    Then by the Pythagorean Theorem, $AB^2+BC^2=CA^2$, so

    $$x^2+10x+100=x^2+6x+36+196,$$

    and

    $$4x=132\Longrightarrow x=033.$$

18. **AIME 1991 #12:** Rhombus $PQRS$ is inscribed in a rectangle $ABCD$ so that the vertices P, Q, R, S are interior points on sides AB, BC, CD, DA, respectively. It is given that $PB = 15$, $BQ = 20$, $PR = 30$, and $QS = 40$. Find the perimeter of $ABCD$.

Solution: The triangles QOB, OBC are isosceles, and similar (because they have $\angle QOB = \angle OBC$). Hence $\frac{BQ}{OB} = \frac{OB}{BC} \Rightarrow OB^2 = BC \cdot BQ$. The length of OB could be found easily from the area of BPQ:

$$BP \cdot BQ = \frac{OB}{2} \cdot PQ \Rightarrow OB = \frac{2BP \cdot BQ}{PQ} \Rightarrow OB = 24,$$

$$OB^2 = BC \cdot BQ \Rightarrow 24^2 = (20 + CQ) \cdot 20 \Rightarrow CQ = \frac{44}{5}.$$

From the right triangle CRQ we have $RC^2 = 25^2 - \left(\frac{44}{5}\right)^2 \Rightarrow RC = \frac{117}{5}$. We could have also defined a similar formula: $OB^2 = BP \cdot BA$, and then we found AP, the segment OB is tangent to the circles with diameters AO, CO. The perimeter is $2(PB + BQ + QC + CR) = 2\left(15 + 20 + \frac{44+117}{5}\right) = \frac{672}{5}$.

19. **AIME 1996 #13** In $\triangle ABC$, $AB = \sqrt{30}$, $AC = \sqrt{6}$, and $BC = \sqrt{15}$. There is a point D for which AD bisects segment BC and that $\angle ADB$ is a right angle. Find the ratio $[ADB] : [ABC]$.

Solution: Let E be the midpoint of $\overline{BC}$. Since $BE = EC$, then $\triangle ABE$ and $\triangle AEC$ share the same height and have equal bases, and thus have the same area. Similarly, $\triangle BDE$ and BAE share the same height, and have bases in the ratio $DE : AE$, so $\frac{[BDE]}{[BAE]} = \frac{DE}{AE}$ (see area ratios). Now,
$\dfrac{[ADB]}{[ABC]} = \frac{[ABE]+[BDE]}{2[ABE]} = \frac{1}{2} + \frac{DE}{2AE}$. By Stewart's Theorem, $AE = \frac{\sqrt{2(AB^2+AC^2)-BC^2}}{2} = \frac{\sqrt{57}}{2}$, and by the Pythagorean Theorem on $\triangle ABD, \triangle EBD$,

$$BD^2 + \left(DE + \frac{\sqrt{57}}{2}\right)^2 = 30$$
$$BD^2 + DE^2 = \frac{15}{4}$$

Subtracting the two equations yields $DE\sqrt{57} + \frac{57}{4} = \frac{105}{4} \implies DE = \frac{12}{\sqrt{57}}$. Then $\frac{m}{n} = \frac{1}{2} + \frac{DE}{2AE} = \frac{1}{2} + \frac{\frac{12}{\sqrt{57}}}{2 \cdot \frac{\sqrt{57}}{2}} = \frac{27}{38}$.

20. **Junior Balkan MO 2009 #1** Let $ABCDE$ be a convex pentagon such that $AB + CD = BC + DE$ and a circle ω with center O on the side AE touches the sides AB,

BC, CD and DE at points P, Q, R and S, respectively. Prove that the lines PS and AE are parallel.

Solution: Combining Equal Tangents from B, C, D with the given condition $AB + CD = BC + DE$ gives $AP = ES$. Hence the triangles OPA and OSE are congruent (HL) implying that the P-height in $\triangle OPA$ equals the S-height in $\triangle OSE$. The result follows.

21. **IMO 1960 #3** In a given right triangle ABC, the hypotenuse BC, of length a, is divided into n equal parts (n an odd integer). Let α be the acute angle subtending, from A, that segment which contains the midpoint of the hypotenuse. Let h be the length of the altitude to the hypotenuse of the triangle. Prove that:

$$\tan\alpha = \frac{4nh}{(n^2-1)a}.$$

Solution: Let P, Q, R be points on side BC such that segment PR contains midpoint Q, with P closer to C and (without loss of generality) $AC \le AB$. Then if AD is an altitude, then D is between P and C. Combined with the obvious fact that Q is the midpoint of PR (for n is odd), we have

$$\begin{aligned}
\tan\angle PAR &= \tan(\angle RAD - \angle PAD) \\
&= \frac{\frac{PR}{h}}{1+\frac{DP\cdot DR}{h^2}} \\
&= \frac{PR\cdot h}{h^2 + DP\cdot DR} \\
&= \frac{PR\cdot h}{AQ^2 - DQ^2 + DP\cdot DR} \\
&= \frac{PR\cdot h}{\frac{a^2}{4} - PQ^2} = \frac{\frac{a}{n}\cdot h}{\frac{a^2}{4}-\frac{a^2}{4n^2}} = \frac{4nh}{(n^2-1)a}.
\end{aligned}$$

22. **IMO 1983 #6** Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2(c-a) \ge 0.$$

Determine when equality occurs.

Solution: By the Dual Principle we can substitute, $a = y+z$, $b = z+x$, $c = x+y$. Then, the triangle condition becomes $x, y, z > 0$. After some manipulation, the inequality becomes

$$xy^3 + yz^3 + zx^3 \ge xyz(x+y+z).$$

By Cauchy-Schwarz, we have:

$$(xy^3 + yz^3 + zx^3)(z+x+y) \ge xyz(y+z+x)^2,$$

with equality if and only if $\frac{xy^3}{z} = \frac{yz^3}{x} = \frac{zx^3}{y}$. So the inequality holds with equality if and only if $x = y = z$. Thus the original inequality has equality if and only if the triangle is equilateral.

Circles and Angles

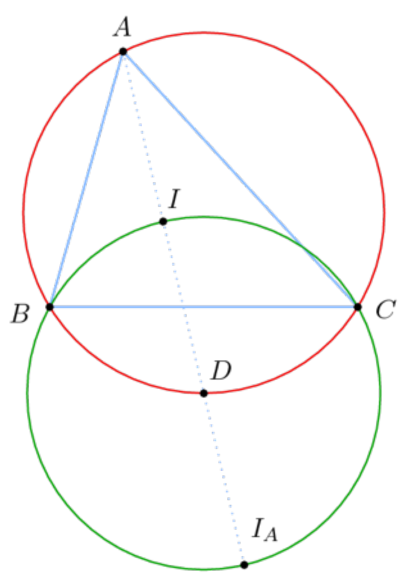
Inscribed Angle Theorem: Let O be the circumcenter of the triangle ABC. Then $\angle AOB = 2 \cdot \angle ACB$. Hence the locus of points X such that $\angle AXB = \angle ACB$ is a union of two circular arcs symmetric about AB.

- Try to prove this on your own! Consider the case where one of the sides of the triangle is the diameter of the circumcircle. Then consider the cases where the circumcenter lies either inside or outside of the triangle. Draw diagrams for each case and angle chase knowing that the sum of the interior angles of a triangle is $180° = \pi$ and that a circle represents $360° = 2\pi$.

Arcs As Angles: Let ω be a circle.

1. Given two equal arcs of ω, the corresponding inscribed angles are the same.
2. Given two equal angles inscribed in (i.e. with the vertices on) ω, the arcs they intercept on ω are of the same length.

Incenter-Excenter Lemma ("Fact Five"): Let ABC be a triangle with incenter I, A-excenter I_A. Denote by D the midpoint of arc BC on the circumcircle. The angle bisector of $\angle A$ and the perpendicular bisector of the side BC intersect at D. Furthermore, D is the center of the circle through I, I_A, B, and C.



This is a very important result! The behind why this theorem is sometimes called “Fact Five” in the Olympiad community is actually a bit of a meme, and we will discuss it in class.[2]
Proof: We proceed with angle-chasing. Let $A = \angle BAC$, $B = \angle CBA$, $C = \angle ACB$. Note that D is where the perpendicular bisector of BC intersects the circumcircle of $\triangle ABC$. We have that A, I, D are collinear, as D is on the angle bisector and is the midpoint of arc BC. We wish to show that $DB = DI$, the other cases being similar.
zvFirst, notice that $\angle DBI = \angle DBC + \angle CBI = \angle DAC + \angle CBI = \angle IAC + \angle CBI = \frac{1}{2}A + \frac{1}{2}B$.
However, $\angle BID = \angle BAI + \angle ABI = \frac{1}{2}A + \frac{1}{2}B$.
Hence, $\triangle BID$ is isosceles. So $DB = DI$.
We can apply a similar strategy for showing $DC = DI_A = DI$.
Cyclic Quadrilaterals: Let $ABCD$ be a quadrilateral. We say $ABCD$ is cyclic, or that it can be inscribed into a circle, if it has the following properties:

1. $\angle A + \angle C = \angle B + \angle D = 180^o$
2. $\angle ABD = \angle ACD$
3. $\angle BCA = \angle BDA$
4. $\angle BAC = \angle BDC$
5. $\angle CAD = \angle CBD$

Ptolemy’s Theorem: Given a cyclic quadrilateral $ABCD$ with side lengths AB, BC, CD, AD and diagonals AC, BD:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Proof: Given cyclic quadrilateral $ABCD$, extend CD to P such that $\angle BAD = \angle CAP$. Since quadrilateral $ABCD$ is cyclic, $m\angle ABC + m\angle ADC = 180°$. However, $\angle ADP$ is also supplementary to $\angle ADC$, so $\angle ADP = \angle ABC$. Hence, $\triangle ABC \sim \triangle ADP$ by AA similarity and

$$\frac{AB}{AD} = \frac{BC}{DP} \implies DP = \frac{(AD)(BC)}{(AB)}.$$

Now, note that $\angle ABD = \angle ACD$ (subtend the same arc) and $\angle BAC + \angle CAD = \angle DAP + \angle CAD \implies \angle BAD = \angle CAP$, so $\triangle BAD \sim \triangle CAP$. This yields

$$\frac{AB}{AC} = \frac{BD}{CP} \implies CP = \frac{(AC)(BD)}{(AB)}.$$

However, $CP = CD + DP$. Substituting in our expressions for CP and DP,

$$\frac{(AC)(BD)}{(AB)} = CD + \frac{(AD)(BC)}{(AB)}.$$

[2] `http://web.evanchen.cc/handouts/Fact5/Fact5.pdf`

Multiplying by AB yields $(AC)(BD) = (AB)(CD) + (AD)(BC)$.

One might be curious about the case where $ABCD$ is *not* cyclic. As it turns out, there is a stronger version of Ptolemy's theorem can applies to any four points in space.

Ptolemy's Inequality: Let A, B, C, D be four points in a plane, we have

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD,$$

with equality if and only if $ABCD$ is a cyclic quadrilateral with diagonals AC and BD. This also holds if A, B, C, D are four points in space not in the same plane, but equality can't be achieved.

Proof (Coplanar Case): We construct a point P such that the triangles APB, DCB are similar and have the same orientation. In particular, this means that

$$BD = \frac{BA \cdot DC}{AP}. \tag{1}$$

But since this is a spiral similarity, we also know that the triangles ABD, PBC are also similar, which implies that

$$BD = \frac{BC \cdot AD}{PC}. \tag{2}$$

Now, by the triangle inequality, we have $AP + PC \geq AC$. Multiplying both sides of the inequality by BD and using (1) and (2) gives us

$$BA \cdot DC + BC \cdot AD \geq AC \cdot BD,$$

which is the desired inequality. Equality holds iff. A, P, and C are collinear. But since the triangles BAP and BDC are similar, this would imply that the angles BAC and BDC are congruent, i.e., that $ABCD$ is a cyclic quadrilateral.

- **Sketch for Three Dimensions:** Construct a sphere passing through the points B, C, D and intersecting segments AB, AC, AD and E, F, G. We can now prove it through similar triangles, since the intersection of a sphere and a plane is always a circle.

- **Motivation for All Dimensions:** Similar to the fact that that there is a line through any two points and a plane through any three points, there is a three-dimensional "solid" or 3-plane through any four points. Thus in an n-dimensional space, one can construct a 3-plane through the four points and the theorem is trivial, assuming the case has already been proven for three dimensions.

- **Proof (All Dimensions):** Let any four points be denoted by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

Note that

$(\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{d})+(\mathbf{a}-\mathbf{d})\cdot(\mathbf{b}-\mathbf{c})$

$=\mathbf{a}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{d}-\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{d}+\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}-\mathbf{d}\cdot\mathbf{b}+\mathbf{d}\cdot\mathbf{c}$

$=\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{d}-\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{d}$

$=(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{d})$.

From the Triangle Inequality,

$|(\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{d})|+|(\mathbf{a}-\mathbf{d})\cdot(\mathbf{b}-\mathbf{c})|\geq|(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{d})|$

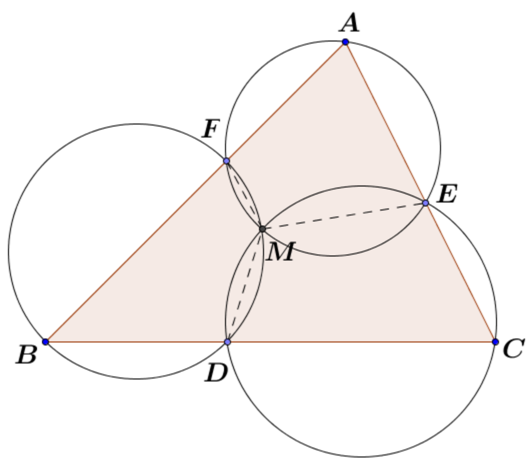
$\implies |\mathbf{a}-\mathbf{b}||\mathbf{c}-\mathbf{d}|+|\mathbf{a}-\mathbf{d}||\mathbf{b}-\mathbf{c}|\geq|\mathbf{a}-\mathbf{c}||\mathbf{b}-\mathbf{d}|$

$\implies AB\cdot CD+AD\cdot BC\geq AC\cdot BD$.

Angle Between Chords: Let $ABCD$ be a cyclic quadrilateral and denote $P=AC\cap BD$ and $Q=AB\cap CD$. Moreover, let β, δ $(\beta>\delta)$ be the inscribed angles corresponding to the arcs BC, DA (not containing other vertices). Then $\angle BPC=\beta+\delta$ and $\angle BQC=\beta-\delta$. The proof is left as an exercise to the reader.

Angle by Tangent: Let ℓ be a line passing through the vertex B of a triangle ABC and for the notation purposes, let L be a point on this line such that BC separates A and L. Then ℓ is tangent to the circumcircle of $\triangle ABC$ (at B) iff $\angle LBC=\angle BAC$. The proof is left as an exercise to the reader.

Miquel's (Pivot) Theorem: Let K, L, M be the points on the sides BC, CA, AB of a triangle ABC, respectively. Prove that the circumcircles of the triangles ALM, BKM, CKL pass through a common point called the **Miquel Point**.



Proof: Consider the above configuration. Suppose that circles BDF and CDE intersect at $M\neq D$. So the quadrilaterals $BDMF$ and $CEMD$ are cyclic, meaning $\angle DMF=180^\circ-\angle B$ and $\angle DME=180^\circ-\angle C$. This gives

$$\angle EMF=360^\circ-(\angle DMF+\angle DME)=\angle B+\angle C.$$

Since $\angle B + \angle C$ also equals $180° - \angle A$ from $\triangle ABC$, the circle passing through the points E, F, and M must also pass through the point A so that the quadrilateral $AEMF$ is also cyclic, i.e. the circle AEF passes through M. So, M is the Miquel point, and the proof is complete.

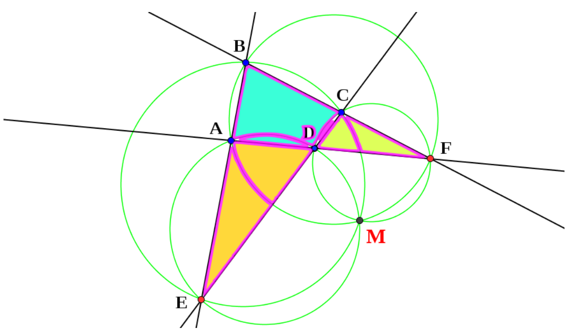
Note that we can easily extend this to several useful results. Try to prove the following results on your own![3]

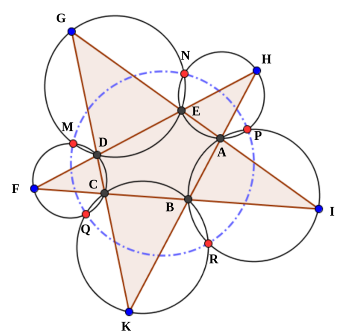
Miquel's Theorems:

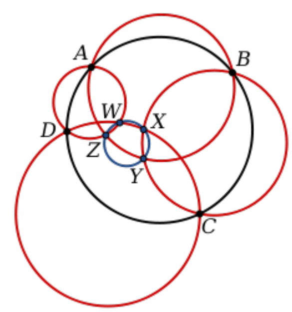
1. **Quadrilateral Theorem:** Four lines form four triangles. The circumscribed circles of these triangles have a common point.

2. **Pentagon Theorem:** Let $ABCDE$ be a convex pentagon. Extend all sides until they meet in five points F, G, H, I, K and draw the circumcircles of the five triangles $\triangle CFD$, $\triangle DGE$, $\triangle EHA$, $\triangle AIB$ and $\triangle BKC$. Then the second intersection points (other than A, B, C, D, E), namely the new points M, N, P, R and Q are concyclic.

3. **Six Circles Theorem:** Given points A, B, C, and D on a circle, and circles passing through each adjacent pair of points, the alternate intersections of these four circles at W, X, Y, and Z then lie on a common circle.







Right Angles Create Cyclic Quads: Let ABC be an acute triangle. Denote by K, L, M the feet of its A-, B-, and C-heights and by H their intersection (i.e. the *orthocenter* of $\triangle ABC$). This creates 6 cyclic quadrilaterals.

Proof: Since $\angle ALH = \angle AMH = 90°$, the quadrilateral $ALHM$ is cyclic and likewise $BKHM$ and $CLHK$ are cyclic. Also, from $\angle BLC = \angle BMC = 90°$ we deduce that $BMLC$ is cyclic and for the same reasons $ALKB$ and $AMKC$ are cyclic too.

Tied Circles: Let $ABCD$ be a cyclic quadrilateral and denote by P its intersection of diagonals. Let circle ω passing through A and B intersect segments PC, PD at X, Y, respectively. We have that that XY is parallel to CD.

[3] `https://en.wikipedia.org/wiki/Miquel%27s_theorem`

Proof: Using both the cyclic quadrilaterals $ABXY$ and $ABCD$ we have $\angle BXY = \angle BAY = \angle BAC = \angle BDC$, so $XY \parallel CD$ indeed.

Shooting Lemma: Let M be the midpoint of arc XY of a circle ω. Chords MA, MB of ω intersect the segment XY at D, C, respectively. We have that $ABCD$ is cyclic.

Proof: It suffices to prove $\angle MAB = \angle MCD$. By angle between chords, the latter one is the sum of inscribed angles corresponding to the arcs MX and YB while the first one is clearly corresponding to MB. We conclude by observing that the arcs $MX = MY$.

Brocard Point: Let ABC be a triangle. Denote by ω_a the circle tangent to AB at A and passing through C. Likewise, let ω_b be the circle tangent to BC at B and passing through A, and ω_c the circle tangent to CA at C and passing through B. The circles ω_a, ω_b, ω_c intersect at one point.

Proof: Denote the intersection of ω_a and ω_b by X. Angles by Tangents give us $\angle CBX = \angle BAX = \angle ACX$. Hence (Angle by Tangent in the other direction) CA is tangent to the circumcircle of triangle BCX, so X lies on ω_a too.

Worked Examples:

1. Let $\triangle ABC$ be an equilateral triangle. Let P be a point on minor arc AB of its circumcircle. Prove that $PC = PA + PB$.

   Solution: Draw PA, PB, PC. By Ptolemy's Theorem applied to quadrilateral $APBC$, we know that $PC \cdot AB = PA \cdot BC + PB \cdot AC$. Since $AB = BC = CA = s$, we divide both sides of the last equation by s to get the result: $PC = PA + PB$.

2. In a regular heptagon $ABCDEFG$, prove that: $\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}$.

   Solution: Let $ABCDEFG$ be the regular heptagon. Consider the quadrilateral $ABCE$. If a, b, and c represent the lengths of the side, the short diagonal, and the long diagonal respectively, then the lengths of the sides of $ABCE$ are a, a, b and c; the diagonals of $ABCE$ are b and c, respectively.

   Now, Ptolemy's Theorem states that $ab + ac = bc$, which is equivalent to $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ upon division by abc.

3. Point P lies inside an acute angle BAC of unknown measure α such that $\angle PAB = 20°$. Drop perpendiculars PK and PL to lines AB and AC. Find a cyclic quadrilateral and prove that $\angle ALK$ doesn't depend on α.

   Solution: Since $\angle PKA = 90° = \angle PLA$, quadrilateral PB_1AC_1 is cyclic and we have $\angle ALK = \angle APK = 180° - 90° - 20° = 70°$.

4. Persuade yourself that if a trapezoid is cyclic then it is isosceles. In other words, realize

that the arcs between two parallel lines are the same.

Solution: If the two lines are both horizontal, then the whole diagram has a vertical axis of symmetry.

Alternatively, let $ABCD$ be a cyclic trapezoid with $AB \parallel CD$. By parallel lines, $\angle BAC = \angle ACD$, so the inscribed angles corresponding to the arcs BC and AD are the same. Hence the arcs are of equal length and as a consequence, $BC = AD$.

5. Segments AC and BD interect at P. Denote the second intersection of the circumcircles of APD and BCD by S. Prove that $\triangle SAD \sim \triangle SCB$.

   Solution: We prove the similarity by AA. Using cyclic quadrilaterals $ASPD$ and $BSPC$ we have $\angle SAD = 180^\circ - \angle DPS = \angle SPB = \angle SCB$ and likewise $\angle ADS = \angle CBS$, hence we are done.

6. Circles ω, ω' intersect at A and B. An arbitrary line through A intersects them for the second time at C, D, respectively. The tangents at C, D to the respective circles intersect at P. Prove that P lies on the circumcircle of $\triangle BCD$.

   Solution: It suffices to prove that $BCPD$ is cyclic. Using the Angle by Tangent, we may write $\angle PCD = \angle CBA$ and $\angle CDP = \angle ABD$. Hence $\angle CPD = 180^\circ - (\angle PCD + \angle CDP) = 180^\circ - \angle CBD$ which finishes the proof.

7. Suppose that in $\triangle ABC$ with circumcenter O, centroid G, and circumradius R, we have $OG = \frac{1}{3}R$. Prove that the triangle is right-angled.

   Solution: Note that by the Euler Line, $OH = 3OG = R$. Thus H lies on the circumcircle of $\triangle ABC$. If the triangle is acute, then the orthocenter must necessarily lie in its interior; if the triangle is obtuse, then the orthocenter must necessarily lie outside of its interior (why?). Therefore $\triangle ABC$ must be right-angled, and the conclusion follows.

8. Let I be the incenter of the triangle ABC. Lines AI, CI intersect the circumcircle ω of $\triangle ABC$ for the second time at X, Z, respectively. Prove that XZ is perpendicular to BI.

   Solution: By Arcs As Angles, X and Z are the midpoints of arcs BC, BA not containing A, C, respectively. Let Y be the intersection of BI and ω (i.e. the midpoint of arc AC). Then the arcs BX and YZ combined form one half of the perimeter of ω, so the angle between chords BY and XZ is $\frac{1}{2}180^\circ = 90^\circ$.

9. Let AC and BD be two perpendicular chords of a circle. Prove that the tangent lines to the circle at A, B, C, D form a cyclic quadrilateral.

   Solution: Rewriting the four tangencies recalling Angle by Tangent we learn that the sum of the opposite angles of that quadrilateral is $360° - 2 \cdot (\angle ABD + \angle BAC) = 360° - 2 \cdot 90° = 180°$, so the quadrilateral is indeed cyclic.

10. Circles ω, ω' intersect at A and B. Let P be a variable point on ω and denote by C, D the second intersections of PA, PB with ω'. Prove that the length of the segment CD does not depend on the position of P..

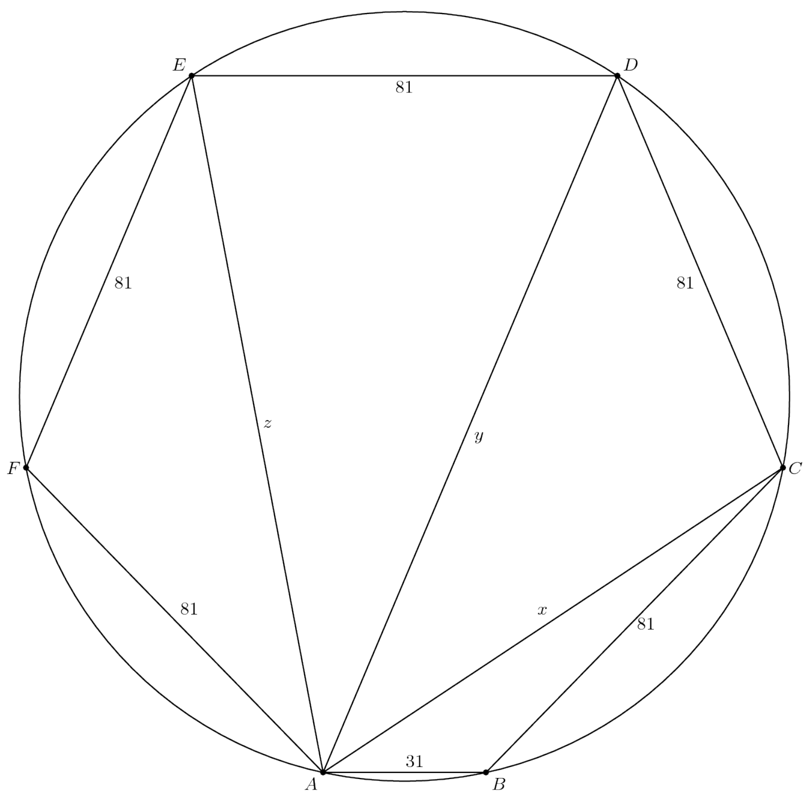
    Solution: Note that $\angle APB$ is fixed and the angle inscribed in ω' corresponding to AB is fixed too. Thus, by angle between chords (difference), the angle inscribed in ω' corresponding to CD is fixed. Hence the length of CD is fixed.

    Another way to write it down is to connect B and C and observe that $\angle CBD = 180° - \angle CBP = \angle APB + \angle ACB$. Since both the summands on the right-hand side are fixed (angles inscribed in ω, ω' corresponding to the fixed arcs AB), the left-hand side is fixed too and we conclude as before.

11. **Sharygin 2012 # 5** On side AC of triangle ABC an arbitrary point is selected D. The tangent in D to the circumcircle of triangle BDC meets AB in point C_1; point A_1 is defined similarly. Prove that $A_1C_1 \parallel AC$.

    Solution: Draw in line segment BD. Note that $\angle BDA_1 = \angle BAD$ and $\angle BDC_1 = \angle BDC$, so $\angle CDA_1 = \angle BAC + \angle BCA = 180° - \angle ABC$. Therefore BA_1DC_1 is cyclic, so $\angle BCA = \angle C_1DB = \angle C_1A_1B$ and $A_1C_1 \parallel AC$ as desired.

12. **AIME 1991 # 14:** A hexagon is inscribed in a circle. Five of the sides have length 81 and the sixth, denoted by $\overline{AB}$, has length 31. Find the sum of the lengths of the three diagonals that can be drawn from A.



Solution: Let $x = AC = BF$, $y = AD = BE$, and $z = AE = BD$. Ptolemy's Theorem on $ABCD$ gives $81y + 31 \cdot 81 = xz$, and Ptolemy on $ACDF$ gives $x \cdot z + 81^2 = y^2$. Subtracting these equations give $y^2 - 81y - 112 \cdot 81 = 0$, and from this $y = 144$. Ptolemy on $ADEF$ gives $81y + 81^2 = z^2$, and from this $z = 135$. Finally, plugging back into the first equation gives $x = 105$, so $x + y + z = 105 + 144 + 135 = 384$.

Angle Chasing

Prove these on your own, then use them forever.

1. **Angles by Important Points:** Let ABC be an acute triangle with incenter I, orthocenter H, and circumcenter O. Prove that $\angle BIC = 90^\circ + \frac{1}{2}\angle A$, $\angle BHC = 180^\circ - \angle A$, and $\angle BOC = 2\angle A$.

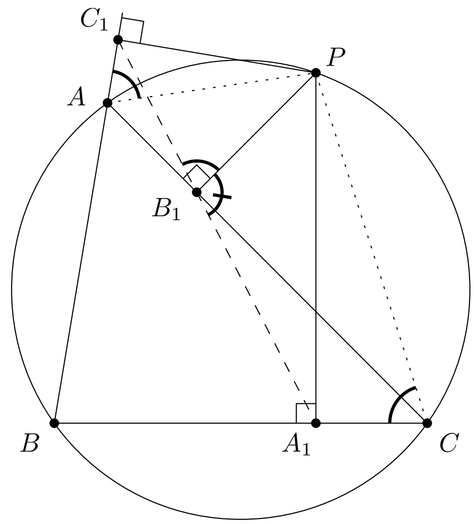
2. **Orthocenter Reflections:** Let H be the orthocenter of a triangle ABC. The reflections of H about the sides of $\triangle ABC$ and about the midpoints of the sides lie on the circumcircle of $\triangle ABC$.

3. **Midpoint of Arc:** Let I, E be the incenter and the E-excenter of a triangle ABC, respectively. Then $BECI$ is cyclic and the center of its circumcircle is the midpoint of arc BC of the circumcircle of $\triangle ABC$ (that does not contain A).

Orthic Triangle: Let the heights AD, BE, CF of $\triangle ABC$ intersect at H. Prove that H is the incenter of triangle DEF.

Proof: It suffices to prove $\angle HDE = \angle HDF$. Using the cyclic quadrilaterals $DCEH$, $DCAF$ we learn that both these angles are in fact equal to $\angle HCA$, so we are done.

Simson Line: Let P be a point on the circumcircle of triangle ABC and denote by A_1, B_1, C_1 the feet of perpendiculars dropped from P onto BC, CA, AB, respectively. Prove that the points A_1, B_1, C_1 are collinear.

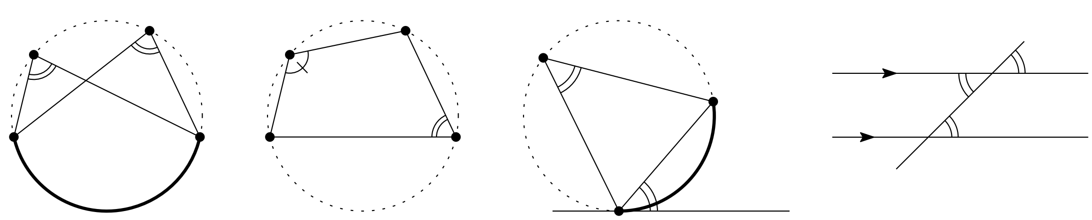


Proof: Notice that PC_1AB_1 is cyclic since $\angle PC_1A + \angle PB_1A = 90° + 90° = 180°$ and PB_1A_1C is cyclic because $\angle PB_1C = \angle PA_1C = 90°$. Since $ABCP$ is also cyclic we write

$$180° - \angle A_1B_1P = \angle A_1CP = \angle BCP = 180° - \angle BAP = \angle C_1AP = \angle C_1B_1P.$$

Hence $\angle A_1B_1C_1 = 180°$, implying that A_1, B_1, C_1 are collinear.

Important Subdiagrams:



Worked Examples:

1. Two circles intersect at points P and Q. Point A is on the first circle. Lines AP and AQ intersect the second circle at points B and C. Prove that the tangent at A to the first circle is parallel to line BC.

   Solution: For the notation purposes, let T be a point on the tangent such that AP separates T and Q. Since AT is a tangent and P, Q, B, C lie on a single circle, we have $\angle TAP = \angle AQP = \angle CBT$. The result follows.

2. Inside square $ABCD$ a point P is taken so that ABP is equilateral. Prove that $\angle PCD = 15°$.

   Solution: Focus on triangle PBC. Since $BP = BA = BC$, the triangle is B-isosceles. Moreover, $\angle CBP = 90° - 60° = 30°$, thus $\angle PCB = 90° - \frac{1}{2}30° = 75°$ meaning that $\angle PCD = 15°$ indeed.

3. Let I, O, H be the incenter, circumcenter, and orthocenter of an acute triangle ABC, respectively. Prove that if the points B, C, H, I lie on a single circle, then O lies on this circle too.

   Solution: Solving $180° - \angle A = \angle BHC = \angle BIC = 90° + \frac{1}{2}\angle A$ yields $\angle A = 60°$. Hence $\angle BHC = \angle BIC = 120°$. Since $\angle BOC = 2\angle A$, it is equal to $120°$ too which finishes the proof.

4. Let $ABCD$ be a cyclic quadrilateral whose diagonals are perpendicular. If M is the midpoint of AB and P is the intersection of AC and BD, then prove that MP is perpendicular to CD.

   Solution: Denote the intersection of MP and CD by X. Observe that the midpoint of AB is the circumcenter of ABP (in particular, triangles MPA, MPB are both M-isosceles). Straightforward angle chasing then gives

   $$\angle XPD = \angle MPB = \angle PBM = 90° - \angle BAC = 90° - BDC.$$

   Hence the sum of two angles in the triangle DXP equals $90°$ implying that the third one is also $90°$.

5. Let $ABCD$ be a rectangle. Point P is inside the rectangle so that $\angle APB + \angle CPD = 180°$. Prove that $\angle ABP + \angle PDC = 90°$.

   Solution: Let P' be the point such that $\triangle AP'B \cong \triangle DPC$. Then $AP'BP$ is cyclic and PP' is perpendicular to AB. Since $\angle PDC = \angle P'AB = \angle P'PB$, we may conclude.

6. Suppose κ_1, κ_2, and κ_3 are three circles intersecting at a common point O. Let A_1 be the point distinct from O at which κ_2 and κ_3 intersect; define A_2 and A_3 similarly. For an arbitrary point $X \in \kappa_1$, let Y denote the second intersection point of XA_3 with ω_2, and let Z be the second intersection point of YA_1 with ω_3. Prove that X, A_2, and Z are collinear.

   Solution: It suffices to prove that $\angle XA_2O + \angle ZA_2O = \pi$. To do this, remark that

   $$\angle XA_2O = \angle YA_3O = \angle ZA_1O = \pi - \angle ZA_2).$$

   This implies the conclusion.

7. Let ABC be a triangle with orthocenter H and circumcircle Γ. Prove that for any point $P \in \Gamma$, the Simson line of P with respect to $\triangle ABC$ bisects the line segment PH.

   Solution: Reword the problem to show that the reflections of P across the side lengths of the triangle lie on a line that passes through H. (The fact that these points lie on a line is trivial.) To do this, let H_a denote the reflection of H across BC, and define H_b and H_c similarly. Note that all these points lie on Γ. Now remark that HP_c is the reflection of PH_c across AB, and similarly HP_a is the reflection of PH_a across BC. Thus, to show that H, P_c, and P_a are collinear is equivalent to showing that $\angle AHP_c = \angle P_aHH_a \implies \angle PH_cP = \angle HH_aP$. But this is obvious, since both angles subtend $\widehat{AP}$. Thus P_a, P_c, and H are collinear. Similar reasoning can be used to show P_b lies on this line as well, so we're done.

8. **AIME I 2009 # 15** In triangle ABC, $AB = 10$, $BC = 14$, and $CA = 16$. Let D be a point in the interior of BC. Let I_B and I_C denote the incenters of triangles ABD and ACD, respectively. The circumcircles of triangles BI_BD and CI_CD meet at distinct points P and D. Find the maximum possible area of $\triangle BPC$.

   Solution: First, by Law of Cosines, we have

$$\cos BAC = \frac{16^2 + 10^2 - 14^2}{2 \cdot 10 \cdot 16} = \frac{256 + 100 - 196}{320} = \frac{1}{2},$$

   so $\angle BAC = 60°$.

   Let O_1 and O_2 be the circumcenters of triangles BI_BD and CI_CD, respectively. We first compute

$$\angle BO_1D = \angle BO_1I_B + \angle I_BO_1D = 2\angle BDI_B + 2\angle I_BBD.$$

   Because $\angle BDI_B$ and $\angle I_BBD$ are half of $\angle BDA$ and $\angle ABD$, respectively, the above expression can be simplified to

$$\angle BO_1D = \angle BO_1I_B + \angle I_BO_1D = 2\angle BDI_B + 2\angle I_BBD = \angle ABD + \angle BDA.$$

Similarly, $\angle CO_2D = \angle ACD + \angle CDA$. As a result

$$\begin{aligned}\angle CPB &= \angle CPD + \angle BPD \\ &= \frac{1}{2} \cdot \angle CO_2D + \frac{1}{2} \cdot \angle BO_1D \\ &= \frac{1}{2}(\angle ABD + \angle BDA + \angle ACD + \angle CDA) \\ &= \frac{1}{2}(2 \cdot 180^\circ - \angle BAC) \\ &= \frac{1}{2} \cdot 300^\circ = 150^\circ.\end{aligned}$$

Therefore $\angle CPB$ is constant (150°). Also, P is B or C when D is B or C. Let point L be on the same side of $\overline{BC}$ as A with $\overline{LC} = \overline{LB} = \overline{BC} = 14$; P is on the circle with L as the center and $\overline{LC}$ as the radius, which is 14. The shortest distance from L to $\overline{BC}$ is $7\sqrt{3}$.

When the area of $\triangle BPC$ is the maximum, the distance from P to $\overline{BC}$ has to be the greatest. In this case, it's $14 - 7\sqrt{3}$. The maximum area of $\triangle BPC$ is

$$\frac{1}{2} \cdot 14 \cdot (14 - 7\sqrt{3}) = 98 - 49\sqrt{3}.$$

9. **Junior Balkan MO 2010 #3** Let AL and BK be angle bisectors in the non-isosceles triangle ABC (L lies on the side BC, K lies on the side AC). The perpendicular bisector of BK intersects the line AL at point M. Point N lies on the line BK such that LN is parallel to MK. Prove that $LN = NA$.

   Solution: Since AL is the bisector of $\angle KAB$, the intersection of AL with the circumcircle of $\triangle AKB$ is the midpoint of minor arc KB. This point also lies on the perpendicular bisector of KB and therefore is M. Hence $AKMB$ is cyclic and $\angle ABK = \angle AMK = \angle ALN$. Therefore $ANLB$ is cyclic. Since BK bisects $\angle ABL$, N is the midpoint of minor arc AL and therefore $LN = NA$.

10. **USAMO 1993 # 2** Let the diagonals of a convex quadrilateral $ABCD$ intersect at P. If $AC \perp BD$, prove that the reflections of P over AB, BC, CD, DA lie on a circle.

    Solution: It suffices to prove that the feet K, L, M, N of perpendiculars dropped from P to AB, BC, CD, DA, respectively lie on a circle. Observe that the segments PK, PL, PM, PN cut $ABCD$ into four cyclic quadrilaterals. Hence

    $$\angle KLM = \angle KLP + \angle PLM = \angle ABP + \angle PCD.$$

    Similarly, $\angle KNM = \angle BAP + \angle PDC$. Thus,

    $$\angle KLM + \angle KNM = (\angle ABP + \angle BAP) + (\angle PCD + \angle PDC) = 90^\circ + 90^\circ = 180^\circ,$$

and we may conclude.

11. **USAMO 2010 # 1** Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle ZOX$, where O is the midpoint of the segment AB.

    Solution: Let N be the foot of perpendicular dropped from Y onto AB. By Simson line (applied for Y and the triangles ABX and ABZ, respectively), both PQ and RS pass through N.

    Since O is the midpoint of diameter, it is the center of the circle and $\angle XOZ$ is the central angle corresponding to the arc XZ, so one half of it is simply the corresponding inscribed angle, i.e. for instance $\angle ZBX$. Hence it suffices to prove that $NBSQ$ is cyclic, which follows from the fact that all Q, N, and S lie on the circle with diameter YB.

12. **All-Russian Olympiad 2011 #2** Given is an acute angled triangle ABC. A circle going through B and the triangle's circumcenter, O, intersects BC and BA at points P and Q respectively. Prove that the intersection of the heights of the triangle POQ lies on line AC.

    Solution: Circumcircles of $\triangle OAQ$ and $\triangle OCP$ meet at O and a point R lying on the sideline AC due to Miquel theorem. Hence, from the cyclic $PORC$ and $QORA$ we have

    $\angle ORP = \angle OCB = \angle OBC\ ,\ \angle ORQ = \angle OAB = \angle OBA \Longrightarrow \angle PRQ = \angle B$

    Since $\angle OPR = \angle OCA = 90^\circ - \angle B$, it follows that $PO \perp RQ$. By similar reasoning, we have $QO \perp RP \Longrightarrow R$ is the orthocenter of $\triangle POQ$.

13. **IMO 2006 # 1** Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

    $$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

    Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

    Solution: We have

    $$\angle IBP = \angle IBC - \angle PBC = \frac{1}{2}\angle ABC - \angle PBC = \frac{1}{2}(\angle PBA - \angle PBC), \qquad (3)$$

    and similarly

$$\angle ICP = \angle PCB - \angle ICB = \angle PCB - \frac{1}{2}\angle ACB = \frac{1}{2}(\angle PCB - \angle PCA). \tag{4}$$

Since $\angle PBA + \angle PCA = \angle PBC + \angle PCB$, we have

$$\angle PBA - \angle PBC = \angle PCB - \angle PCA. \tag{5}$$

By (3), (4), and (5), we get $\angle IBP = \angle ICP$; hence B, I, P, C are concyclic.

Let ray AI meet the circumcircle of ΔABC at point J. Then, by the Incenter-Excenter Lemma, $JB = JC = JI = JP$.

Finally, $AP + JP \geq AJ = AI + IJ$ (since triangle APJ can be degenerate, which happens only when $P = I$), but $JI = JP$; hence $AP \geq AI$ and we are done.

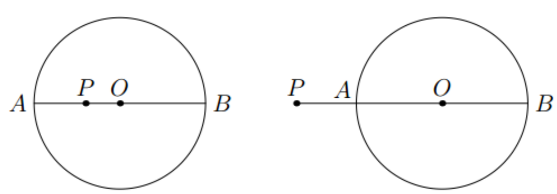
Power of a Point

Power of a Point: Let P be a point and ω a circle. Let ℓ be an arbitrary line passing through P and intersecting ω at A, B, respectively. Then the quantity $PA \cdot PB$ does not depend on the choice of ℓ.

If we denote the center of ω by O and its radius by R then $PA \cdot PB = |OP^2 - R^2|$. The quantity

$$p(P, \omega) = OP^2 - R^2$$

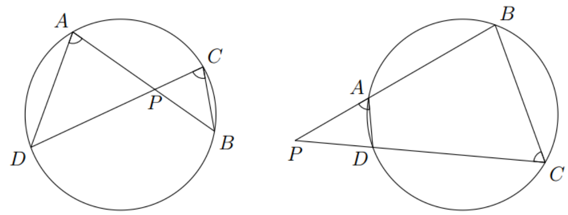
is called the *power of point P with respect to circle ω*. Hence the points inside ω have negative power to it, and the points outside of ω have positive power to it. Also, the locus of points which have given power p to ω is a circle concentric with ω.



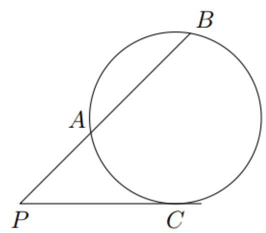
Prove these on your own, then use them forever.

1. Let P be the intersection of diagonals of convex quadrilateral $ACBD$. Then $ACBD$ is cyclic if and only if $PA \cdot PB = PC \cdot PD$.

2. Let P be the intersection of opposite sides AB, CD of a convex quadrilateral $ABCD$. Then $ABCD$ is cyclic if and only if $PA \cdot PB = PC \cdot PD$.



3. Let ABC be a triangle and P a point on the line AB outside of the segment AB. Then PC is tangent to the circumcircle of $\triangle ABC$ if and only if $PC^2 = PA \cdot PB$.



4. If P is outside of ω, then $OP^2 - R^2 = PT^2$, where T is the point where line through P is tangent to ω.

5. Let A be a point outside of a circle ω. Consider all the diameters BC of ω. Prove that the circumcenters of all the triangles ABC lie on a single line.

6. Let Q be the intersection of the lines AD, BC of a cyclic quadrilateral $ABCD$ inscribed in a circle ω. Let a line through Q parallel to AC intersect BD at M. If T is the point on ω such that MT is tangent to ω, prove that $MT = MQ$.

7. Let ABC be a triangle. Line tangent to its circumcircle ω at A intersects BC at P. Denote by M the midpoint of AP and by Q the second intersection of MB and ω. Prove that $\angle PQA = \angle AQC$.

Distance between the Circumcenter and Incenter: Let O be the circumcenter (the point equidistant from the vertices of the triangle) and I be the incenter (the center of the incircle of $\triangle ABC$). Let R be the circumradius and r the inradius. We have $OI^2 = R^2 - 2Rr$.

Proof: Let AI intersect the circumcircle of $\triangle ABC$ at M. Recall that $MI = MB = MC$ by Fact 5. Thus, $(R + OI)(R - OI) = AI \cdot IM$ and we are done.

Common Chord Bisects Tangent: Two circles intersect at A and B. Let MN be their common tangent (with M, N on the respective circles). Prove that AB bisects segment MN.

Proof: Denote the intersection by P. Powers of P with respect to the respective circles yield $PM^2 = PA \cdot PB = PN^2$ and the result follows

Cyclocevian Conjugate: Let ABC be a triangle and P a point inside it. Lines AP, BP, CP meet the opposite sides at D, E, F, respectively. The circumcircle of triangle DEF intersects them for the second time at D', E', F'. Prove that AD', BE', CF' pass through a common point.

Proof: By Ceva's Theorem (to be covered next week),

$$\frac{FB}{BD} \cdot \frac{DC}{CE} \cdot \frac{EA}{AF} = 1.$$

Since

$$\frac{FB}{BD} = \frac{D'B}{BF'}$$

by Power of a Point, we can replace the ratios in the first equality by ratios involving points D', E', F'. In this way we obtain

$$\frac{D'B}{BF'} \cdot \frac{F'A}{AE'} \cdot \frac{E'C}{CD'} = 1$$

and the result follows by the second direction of Ceva's Theorem.

Worked Examples:

1. Let circles ω, ω' intersect at A, B and let P be a point on the segment AB. Line l_1 through P intersects ω at K and L, and line l_2 through P intersects ω' at M and N. Prove that the points K, L, M, N lie on a circle.

   Solution: By Power of P with respect to ω, ω', respectively, we get $PK \cdot PL = PA \cdot PB = PM \cdot PN$. By converse of Power of a Point, points K, L, M, N lie on a single circle.

2. Circle ω and points A, B outside the circle are given. For each line l that passes through A and intersects ω at M and N, consider the circumcircle of $\triangle BMN$. Prove that (depending on the initial configuration) either all these circles have a common point other than B or they are all tangent to the same line at the same point.

   Solution: Denote the second intersection of AB and the circumcircle ω' of $\triangle BMN$ by X. Power of A with respect to ω' is then equal to both

   $$AB \cdot AX = p(A, \omega') = AM \cdot AN.$$

   Since $AM \cdot AN = p(A, \omega)$ is fixed and AB is fixed, the length AX is fixed too. If $AX = AB$ then all the circles ω' are tangent to the fixed line AB at B, otherwise X is the fixed point we were looking for.

3. On the extension of chord KL of a circle centered at O, a point A is taken and tangents AP and AQ to the circle are drawn from it. Let M be the midpoint of PQ. Prove

that $\angle MKO = \angle MLO$.

Solution: Since $AP = AQ$, the median AM is also the perpendicular bisector of PQ. But the perpendicular bisector of any chord passes through the center of the circle, so O, M, A are collinear. Note that

$$\triangle OPA \sim \triangle PMA \quad \Longrightarrow \quad \frac{AO}{AP} = \frac{AP}{AM} \quad \Longrightarrow \quad AM \cdot AO = AP^2.$$

Also, by Power of a Point, $AP^2 = AL \cdot AK$. So $AM \cdot AO = AL \cdot AK$, and by the converse of Power of a Point, $KMOL$ is cyclic. It follows that $\angle MKO = \angle MLO$. (Note: another interesting fact is that if L is further from A than K is, LA is the *symmedian* of $\triangle LPQ$. That is, $\angle PLA = \angle QLM$.

4. **USAMO 1998 # 2** Circle ω is tangent to the segment AC at its midpoint B. Let D be the midpoint of AB. A line passing through A intersects ω at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AC. Find, with proof, the ratio AM/MC.

   Solution: Clearly $AB = BC$ and we may forget the circle ω_1. Then

   $$AE \cdot AF = AB^2 = \left(\frac{1}{2}AB\right) \cdot (2 \cdot AB) = AD \cdot AC$$

   and therefore $CDEF$ is cyclic. Since M is the intersection of perpendicular bisectors of two (nonparallel) sides of a cyclic quadrilateral, it is its circumcenter. Thus $DM = MC$. The rest is straightforward (the answer is $\frac{5}{3}$).

5. **IMO 2009 # 2** Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

   Solution: By Angle by Tangent and recalling that KM is the midline in $\triangle BPQ$ (hence parallel to the base BQ) we may write

   $$\angle KLM = \angle KMQ = \angle PQA.$$

   Likewise, we learn $\angle LKM = \angle QPA$, so $\triangle KLM \sim \triangle PQA$ which rewrites as

   $$\frac{KM}{LM} = \frac{PA}{QA}$$

   or (recalling that midline is half the length of the corresponding side) $\frac{1}{2}BQ \cdot QA = \frac{1}{2}CP \cdot PA$. Hence P and Q have the same power with respect to the circumcircle of $\triangle ABC$, so they are equidistant from its center.

General Notation and Useful Subdiagrams

- A, B, C — "Points A, B, and C"
- $\angle A$, $\angle B$, $\angle C$ — "Angles $\angle BAC$, $\angle ABC$, and $\angle ACB$"
- AB — "Line AB"
- $AB \parallel CD$ — "Line AB is parallel to line CD"
- $AB \perp CD$ — "Line AB is perpendicular to line CD"
- $\triangle ABC$ — "Triangle ABC"
- $\triangle ABC \sim \triangle DEF$ — "Triangle ABC is similar to triangle DEF"
- $\triangle ABC \cong \triangle DEF$ — "Triangle ABC is congruent to triangle DEF"
- $[ABC]$ — "Area of triangle ABC."
- O — The Circumcenter

  The intersection of the perpendicular bisectors of a triangle
- I — The Incenter

  The intersection of the angle bisectors of a triangle
- I_A, I_B, I_C — The A-Excenter, B-Excenter, and C-Excenter

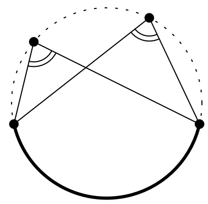
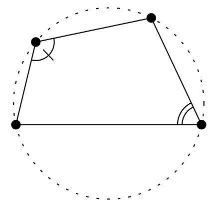
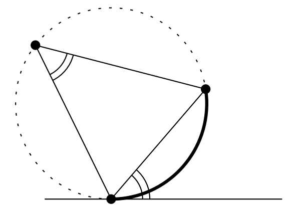
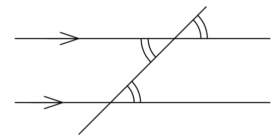
  De ned in the notes
- H — The Orthocenter

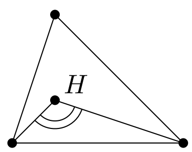
  The intersection of the altitudes of a triangle
- G (sometimes C) — The Centroid

  The intersection of the medians of a triangle
- r — The Inradius

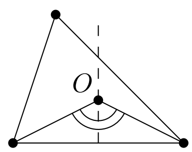
  The radius of the incircle
- R — The Circumradius

  The radius of the circumcircle

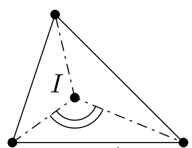







$180^\circ - \angle A$



$2\angle A$



$90^\circ + \frac{1}{2}\angle A$