Geometry Day 1 Problem Set

Math Circle Competition Team

October 8th, 2017

1. Find the inradius and circumradius of the triangle with the sides 13, 14, 15.

   (Hint: Heron's Formula)

2. A hexagon with sides of lengths 2, 2, 7, 7, 11, and 11 is inscribed in a circle. Find the diameter of the circle.

   (Hint: Ptolemy's Theorem)

3. A circle passing through the vertices A, B of a triangle ABC intersects its sides AC, BC for the second time at L, K, respectively. Given that $AL = 2$, $LC = 6$, and $CK = 4$, find KB.

   (Hint: Power of a Point)

4. Circles ω, ω' intersect at A and B. Line tangent to ω passing through A intersects ω' for the second time at X. Likewise, tangent to ω' at B meets ω for the second time at Y. Denote the intersection of the tangents by Z. Given that $BY = 3$ and $YZ = 1$, find AX.

   (Hint: Power of Z with respect to ω')

5. In a triangle ABC there are two points P and Q on side AC, such that $AP > AQ$. The lines BP and BQ divide the median AM into three equal parts. It is known that $PQ = 3$. Find the length of side AC.

   (Hint: Ratio Lemma)

6. Consider triangle ABC with length of $AB = 21$, BD a bisector of angle ABC, $BD = 8\sqrt{7}$, and $DC = 8$. Find the perimeter of ABC.

   (Hint: Angle Bisector Theorem)

7. A circle meets the sides of an equilateral triangle ABC at six points D, E, F, G, H, and I, where D and E are on $\overline{AC}$ with D between E and C, F and G are on $\overline{AB}$ with F between A and G, and H and I are on $\overline{BC}$ with H between B and I. If $AE = 4, ED = 26, DC = 2, FG = 14$, and the circle with diameter HI has area πb, find b.

   (Hint: Power of a Point three times)

8. In triangle ABC, $AB = AC = 100$, and $BC = 56$. Circle P has radius 16 and is tangent to $\overline{AC}$ and $\overline{BC}$. Circle Q is externally tangent to P and is tangent to $\overline{AB}$ and $\overline{BC}$. No point of circle Q lies outside of $\triangle ABC$. The radius of circle Q can be expressed in the form $m - n\sqrt{k}$, where m, n, and k are positive integers and k is the product of distinct primes. Find $m + nk$.

   (Hint: Write BC as the sum of three lengths and calculate two of them with trigonometry)

9. In triangle ABC, the medians AD and CE have lengths 18 and 27, respectively, and $AB = 24$. Extend CE to intersect the circumcircle of ABC at F. Determine the area of triangle AFB.

   (Hint: Power of a Point)

10. **AMC 12B 2008 #11** A cone-shaped mountain has its base on the ocean floor and has a height of 8000 feet. The top $\frac{1}{8}$ of the volume of the mountain is above water. What is the depth of the ocean at the base of the mountain, in feet?

    (Hint: The volume of a cone is $V = \pi r^2 \frac{h}{3}$, how does volume vary with height?)

11. **AMC 12 2012A # 18** Triangle ABC has $AB = 27$, $AC = 26$, and $BC = 25$. Let I denote the incenter of $\triangle ABC$. What is BI?

    (Hint: Heron's Formula)

12. **AMC 12A 2009 #22** A regular octahedron has side length 1. A plane parallel to two of its opposite faces cuts the octahedron into the two congruent solids. What is the area of the polygon formed by the intersection of the plane and the octahedron?

    (Hint: The plane cuts a hexagon, and the sides of the hexagon are midlines in the triangular faces of the octahedron)

13. **AMC 10B 2004 #24** In triangle ABC we have $AB = 7$, $AC = 8$, $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of AD/CD?

    (Hint: Fact 5 and Ptolemy's Theorem)

14. **CHMMC Spring 2012 #10** In triangle ABC, the angle bisector from A and the perpendicular bisector of BC meet at point D, the angle bisector from B and the perpendicular bisector of AC meet at point E, and the perpendicular bisectors of BC and AC meet at point F. Given that $\angle ADF = 5°$, $\angle BEF = 10°$, and $AC = 3$, find the length of DF.

    (Hint: Fact 5 then Extended Law of Sines)

15. **HMMT 2007 Geometry # 6** Triangle ABC has $\angle A = 90^\circ$, $AB > AC$, $BC = 25$, and area 150. Circle ω is inscribed in ABC and touches AC at M. Line BM meets ω again at L. Find the length of segment BL.

    (Hint: Drop the altitude from A to BC, which triangles are similar?)

Challenge Problems:

1. Let ω, ω' be two circles intersecting at A and B. Let ℓ be an arbitrary line passing through A and denote its second intersections with ω, ω' by C, D, respectively. For what line ℓ is the length CD maximal?

2. Let K, L, M, N be the midpoints of arcs AB, BC, CD, DA of the circumcircle of a cyclic quadrilateral $ABCD$. Prove that KM is perpendicular to LN.

    (Hint: Angle chase a lot to find which angles are equal to each other)