Geometry Day 1 Problem Set Solutions

Math Circle Competition Team

October 8th, 2017

1. Find the inradius and circumradius of the triangle with the sides 13, 14, 15.

   Solution: Since $s = \frac{1}{2}(13+14+15) = 21$, Heron's formula implies $[ABC] = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. Hence from $[ABC] = sr$ we deduce $r = \boxed{4}$ and from $[ABC] = \frac{abc}{4R}$ we get $R = \boxed{\frac{65}{8}}$.

2. A hexagon with sides of lengths 2, 2, 7, 7, 11, and 11 is inscribed in a circle. Find the diameter of the circle.

   Solution: Consider half of the circle, with the quadrilateral $ABCD$, AD being the diameter. $AB = 2$, $BC = 7$, and $CD = 11$. Construct diagonals AC and BD. Notice that these diagonals form right triangles. You get the following system of equations:

   $(AC)(BD) = 7(AD) + 22$ (Ptolemy's Theorem)

   $(AC)^2 = (AD)^2 - 121$

   $(BD)^2 = (AD)^2 - 4$

   Solving gives $AD = \boxed{14}$.

3. A circle passing through the vertices A, B of a triangle ABC intersects its sides AC, BC for the second time at L, K, respectively. Given that $AL = 2$, $LC = 6$, and $CK = 4$, find KB.

   Solution: By Power of a Point, $CK \cdot CB = CL \cdot CA$. Hence $4(4 + KB) = 6 \cdot 8$ and $KB = \boxed{8}$.

4. Circles ω, ω' intersect at A and B. Line tangent to ω passing through A intersects ω' for the second time at X. Likewise, tangent to ω' at B meets ω for the second time at Y. Denote the intersection of the tangents by Z. Given that $BY = 3$ and $YZ = 1$, find AX.

   Solution: By power of Z with respect to ω' we learn $ZA^2 = ZY \cdot ZB = 1 \cdot 4$, hence $ZA = 2$. Thus, using power of Z with respect to ω gives us $4^2 = 2(2 + AX)$ implying that $AX = \boxed{6}$.

5. In a triangle ABC there are two points P and Q on side AC, such that $AP > AQ$. The lines BP and BQ divide the median AM into three equal parts. It is known that $PQ = 3$. Find the length of side AC.

   Solution: Note that the intersection of AM and BP is the centroid of ABC. Apply Ratio Lemma twice to get $AC = \boxed{10}$.

6. Consider triangle ABC with length of $AB = 21$, BD a bisector of angle ABC, $BD = 8\sqrt{7}$, and $DC = 8$. Find the perimeter of ABC.

   Solution: Let $x = AD$, so by the angle bisector theorem, $BC = \frac{105}{x}$. Use stewart's theorem and factor out $(x+8)$ on both sides to get a quadratic with solution $x = 7$, giving a perimeter of $\boxed{60}$.

7. A circle meets the sides of an equilateral triangle ABC at six points D, E, F, G, H, and I, where D and E are on $\overline{AC}$ with D between E and C, F and G are on $\overline{AB}$ with F between A and G, and H and I are on $\overline{BC}$ with H between B and I. If $AE = 4, ED = 26, DC = 2, FG = 14$, and the circle with diameter HI has area πb, find b.

   Solution: Applying power of a point three times and solving the resulting quadratic gives $HI = 2\sqrt{88}$, so $b = \boxed{88}$.

8. In triangle ABC, $AB = AC = 100$, and $BC = 56$. Circle P has radius 16 and is tangent to $\overline{AC}$ and $\overline{BC}$. Circle Q is externally tangent to P and is tangent to $\overline{AB}$ and $\overline{BC}$. No point of circle Q lies outside of $\triangle ABC$. The radius of circle Q can be expressed in the form $m - n\sqrt{k}$, where m, n, and k are positive integers and k is the product of distinct primes. Find $m + nk$.

   Solution: Write BC as a sum of three lengths, two of which can be calculated using basic trigonometry, giving an answer of $\boxed{254}$.

9. In triangle ABC, the medians AD and CE have lengths 18 and 27, respectively, and $AB = 24$. Extend CE to intersect the circumcircle of ABC at F. Determine the area of triangle AFB.

   Solution: Apply Power of a Point. The answer is $\boxed{8\sqrt{55}}$.

10. **AMC 12B 2008 #11** A cone-shaped mountain has its base on the ocean floor and has a height of 8000 feet. The top $\frac{1}{8}$ of the volume of the mountain is above water. What is the depth of the ocean at the base of the mountain, in feet?

    Solution: In a cone, radius and height each vary inversely with increasing height (i.e. the radius of the cone formed by cutting off the mountain at 4,000 feet is half that of the original mountain). Therefore, volume varies as the inverse cube of increasing height (expressed as a percentage of the total height of cone): $V_I \times \text{Height}^3 = V_N$. Plugging in our given condition, $\frac{1}{8} = \text{Height}^3 \Rightarrow \text{Height} = \frac{1}{2}$, so the depth is $\frac{1}{2} \cdot 8000 = \boxed{4000}$. ft.

11. **AMC 12 2012A # 18** Triangle ABC has $AB = 27$, $AC = 26$, and $BC = 25$. Let I denote the incenter of $\triangle ABC$. What is BI?

Solution: Inscribe circle C of radius r inside triangle ABC so that it meets AB at Q, BC at R, and AC at S. Note that angle bisectors of triangle ABC are concurrent at the center O (also I) of circle C. Let $x = QB$, $y = RC$ and $z = AS$. Note that $BR = x$, $SC = y$ and $AQ = z$. Hence $x + z = 27$, $x + y = 25$, and $z + y = 26$. Subtracting the last 2 equations we have $x - z = -1$ and adding this to the first equation we have $x = 13$.

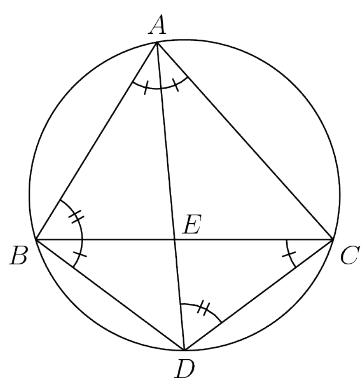
By Heron's formula for the area of a triangle we have that the area of triangle ABC is $\sqrt{39(14)(13)(12)}$. On the other hand the area is given by $(1/2)25r + (1/2)26r + (1/2)27r$. Then $39r = \sqrt{39(14)(13)(12)}$ so that $r^2 = 56$.

Since the radius of circle O is perpendicular to BC at R, we have by the pythagorean theorem $BO^2 = BI^2 = r^2 + x^2 = 56 + 169 = 225$ so that $BI = \boxed{15}$.

12. **AMC 12A 2009 #22** A regular octahedron has side length 1. A plane parallel to two of its opposite faces cuts the octahedron into the two congruent solids. What is the area of the polygon formed by the intersection of the plane and the octahedron?

    Solution: The plane passes through the six midpoints of the corresponding sides, so the plane cuts a hexagon. Every side of this hexagon is a midline in some equilateral triangular face of the given octahedron, so it has the length $\frac{1}{2}$. Since the octahedron possesses central symmetry as well as 3-wise rotational symmetry, the hexagon is regular. Its area is thus simply $6 \cdot \frac{\sqrt{3}}{4}\left(\frac{1}{2}\right)^2 = \boxed{\frac{3}{8}\sqrt{3}}$.

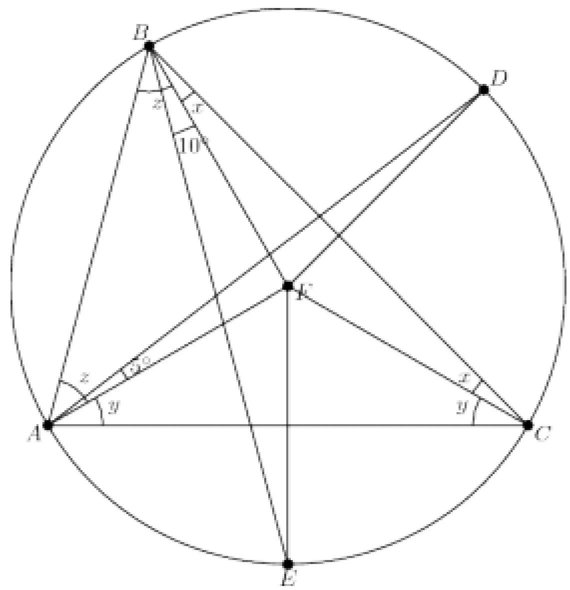
13. **AMC 10B 2004 #24** In triangle ABC we have $AB = 7$, $AC = 8$, $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of AD/CD?



    Solution: Set $\overline{BD}$'s length as x. CD's length must also be x since $\angle BAD$ and $\angle DAC$ intercept arcs of equal length(because $\angle BAD = \angle DAC$). Using Ptolemy's Theorem, $7x + 8x = 9(AD)$. The ratio is $\boxed{\frac{5}{3}}$.

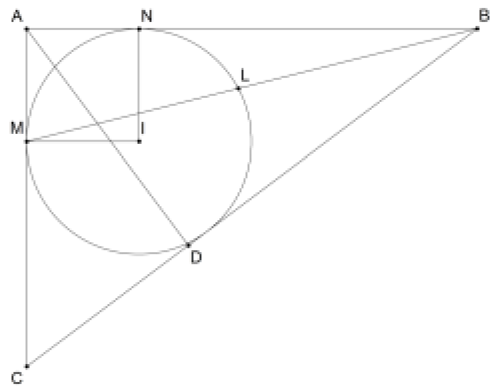
14. **CHMMC Spring 2012 #10** In triangle ABC, the angle bisector from A and the perpendicular bisector of BC meet at point D, the angle bisector from B and the

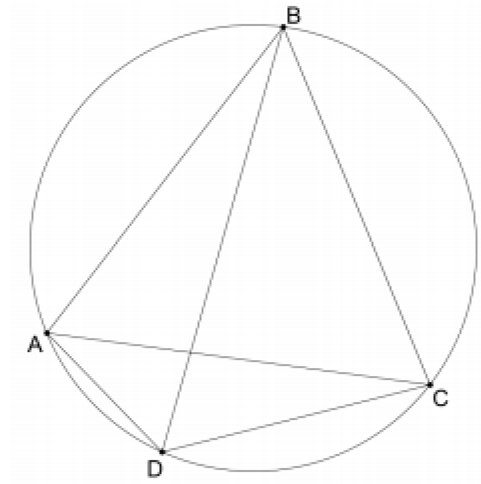
perpendicular bisector of AC meet at point E, and the perpendicular bisectors of BC and AC meet at point F. Given that $\angle ADF = 5^\circ$, $\angle BEF = 10^\circ$, and $AC = 3$, find the length of DF.



Solution: Notice that F is the circumcenter of triangle ABC, that D is the midpoint of arc BC on the circumcircle, and E is the midpoint of arc AC (Fact 5). Hence FA, FB, FC, FD, FE are all radii of the circumcircle and equal. This makes $\triangle ADF$ and $\triangle BEF$ isosceles, so $\angle ADF = \angle DAF = 5^\circ$ and $\angle BEF = \angle EBF = 10^\circ$. Let $\angle FBC = \angle FCB = x$, $\angle FCA = \angle FAC = y$, $\angle FAB = \angle FBA = z$. Using the various angles we know we can find that $x + y + z = 90^\circ$, $y + 5^\circ = z5^\circ$, and $x + 10^\circ = z10^\circ$. Solving this system gives $(x, y, z) = (20^\circ, 30^\circ, 40^\circ)$, so in particular $\angle ABC = 60^\circ$. By the extended law of sines, $\frac{AC}{\sin \angle ABC} = 2R = 2 \cdot DF$. Therefore, $DF = \frac{3}{2 \sin 60^\circ} = \boxed{\sqrt{3}}$.

15. **HMMT 2007 Geometry # 6** Triangle ABC has $\angle A = 90^\circ$, $AB > AC$, $BC = 25$, and area 150. Circle ω is inscribed in ABC and touches AC at M. Line BM meets ω again at L. Find the length of segment BL.





Solution: Let D be the foot of the altitude from A to side BC. The length of AD is $2\frac{150}{25} = 12$. Triangles ADC and BDA are similar, so $CD \cdot DB = AD^2 = 144 \implies BD = 16$ and $CD = 9 \implies AB = 20$ and $AC = 15$. Using equal tangents or the formula $[ABC] = rs$, we can find the radius of ω to be 5. Now, let N be the tangency point of ω on AB. By power of a point, we have $BL \cdot BM = BN^2$. Since the center of ω together with M, A, and N determines a square, $BN = 15$ and $BM = 5\sqrt{17}$, and we have $BL = \boxed{45\frac{\sqrt{17}}{17}}$.

Challenge Problems:

1. Let ω, ω' be two circles intersecting at A and B. Let ℓ be an arbitrary line passing through A and denote its second intersections with ω, ω' by C, D, respectively. For what line ℓ is the length CD maximal?

   Solution: Divert to Aaron

2. Let K, L, M, N be the midpoints of arcs AB, BC, CD, DA of the circumcircle of a cyclic quadrilateral $ABCD$. Prove that KM is perpendicular to LN.

   Solution: Divert to Aaron