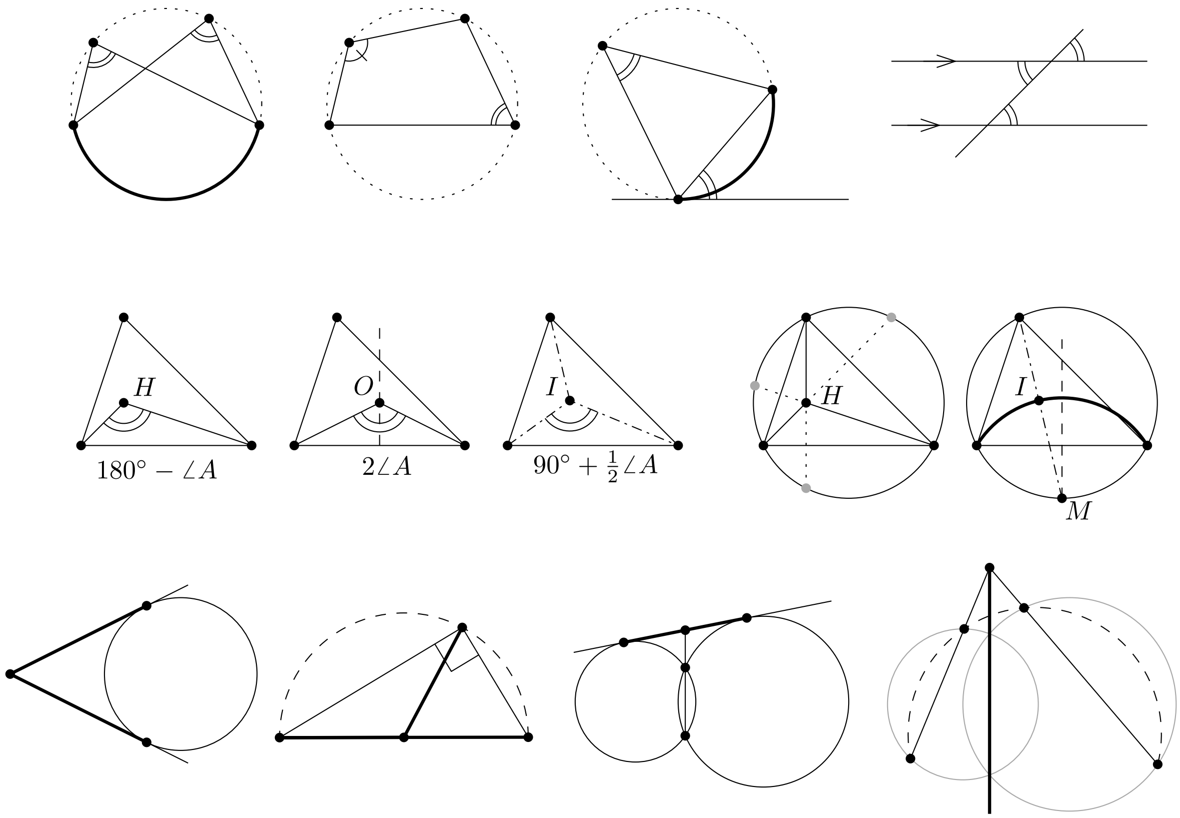
Geometry Day 2 Notes

Math Circle Competition Team

October 15th, 2017

Important Subdiagrams



Ceva and Menelaus

Consider a point on a sideline of a triangle and consider the line from the vertex opposite the sideline to the aforementioned point. We call such a line a *cevian*. Define an *interior* point on a line XY as a point Z such that $Z \in [XY]$ and $Z \neq X$, $Z \neq Y$ (a short notation here is $X * Z * Y$ which notation is read as "Z is between X and Y"). Now suppose $\triangle ABC$ is a triangle. Let A_1, B_1 and C_1 be points on the lines BC, CA and AB respectively, and that none of the three points is a vertex of that triangle. Consider the quantity

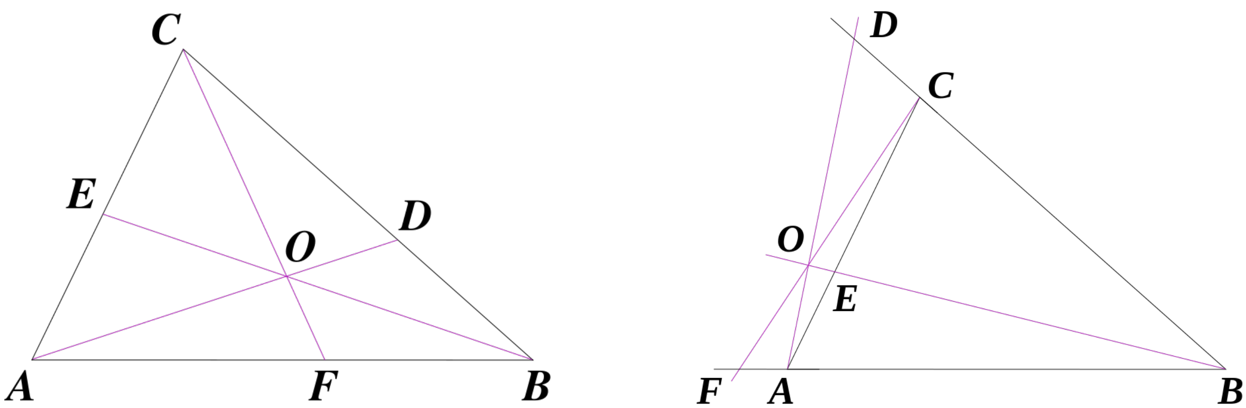
$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = P.$$

We are primarily interested in the cases where $|P| = 1$, as it gives us a nice ratio that leads to several very important results.

Ceva's Theorem: Given a triangle ABC, let the lines AO, BO, and CO be drawn from the vertices to a common point O (not on one of the sides of ABC), to meet opposite sides at D, E, and F respectively. We have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Conversely, if points D, E and F are chosen on BC, AC, and AB respectively so that the above equality is true, then AD, BE, and CF are concurrent, or all three are parallel. The converse is often included as part of the theorem. Note from the following diagrams that the cevians do not necessarily intersect within the triangle. This is not an issue, as we can generalize distance in such configurations with the notion of directed segments.



Proof: We will use the notation $[ABC]$ to denote the area of a triangle with vertices A, B, C. First, suppose AD, BE, CF meet at a point X. We note that triangles ABD, ADC have the same altitude to line BC, but bases BD and DC. It follows that $\frac{BD}{DC} = \frac{[ABD]}{[ADC]}$. The same is true for triangles XBD, XDC, so

$$\frac{BD}{DC} = \frac{[ABD]}{[ADC]} = \frac{[XBD]}{[XDC]} = \frac{[ABD] - [XBD]}{[ADC] - [XDC]} = \frac{[ABX]}{[AXC]}.$$

Similarly, $\frac{CE}{EA} = \frac{[BCX]}{[BXA]}$ and $\frac{AF}{FB} = \frac{[CAX]}{[CXB]}$, so

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{[ABX]}{[AXC]} \cdot \frac{[BCX]}{[BXA]} \cdot \frac{[CAX]}{[CXB]} = 1.$$

Now, suppose D, E, F satisfy Ceva's criterion, and suppose AD, BE intersect at X. Suppose the line CX intersects line AB at F'. We have proven that F' must satisfy Ceva's criterion. This means that $\frac{AF'}{F'B} = \frac{AF}{FB}$, so $F' = F$, and line CF concurrs with AD and BE. ■

Two other common methods of proving Ceva's Theorem involve Routh's Theorem and

Barycentric coordinates. However, both are computationally intensive and beyond the scope of the course. Thus, they have been omitted from this notes packet (but the author strongly encourages any user to look into both topics on their own).

Trigonometric Version of Ceva's Theorem: The trigonometric form of Ceva's Theorem (Trig Ceva) states that cevians AD, BE, CF concur if and only if

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin CBE}{\sin EBA} \cdot \frac{\sin ACF}{\sin FCB} = 1.$$

Proof: First, suppose AD, BE, CF concur at a point X. We note that

$$\frac{[BAX]}{[XAC]} = \frac{\frac{1}{2}AB \cdot AX \cdot \sin BAX}{\frac{1}{2}AX \cdot AC \cdot \sin XAC} = \frac{AB \cdot \sin BAD}{AC \cdot \sin DAC},$$

and similarly,

$$\frac{[CBX]}{[XBA]} = \frac{BC \cdot \sin CBE}{BA \cdot \sin EBA}; \ \frac{[ACX]}{[XCB]} = \frac{CA \cdot \sin ACF}{CB \cdot \sin FCB}.$$

It follows that

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin CBE}{\sin EBA} \cdot \frac{\sin ACF}{\sin FCB} = \frac{AB \cdot \sin BAD}{AC \cdot \sin DAC} \cdot \frac{BC \cdot \sin CBE}{BA \cdot \sin EBA} \cdot \frac{CA \cdot \sin ACF}{CB \cdot \sin FCB},$$

$$= \frac{[BAX]}{[XAC]} \cdot \frac{[CBX]}{[XBA]} \cdot \frac{[ACX]}{[XCB]} = 1. \blacksquare$$

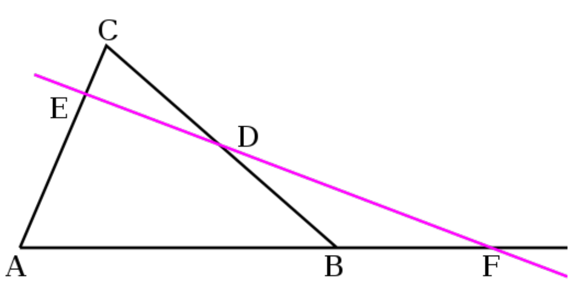
Here, sign is irrelevant, as we may interpret the sines of directed angles mod π to be either positive or negative.[1]

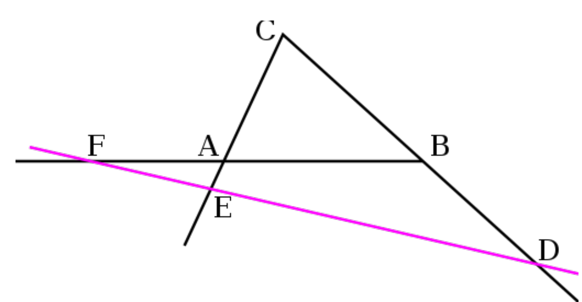
Menelaus' Theorem: Given a triangle ABC, and a transversal line that crosses BC, AC, and AB at points D, E, and F respectively, with D, E, and F distinct from A, B, and C, then we have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1,$$

or simply

$$AF \times BD \times CE = -FB \times DC \times EA.$$

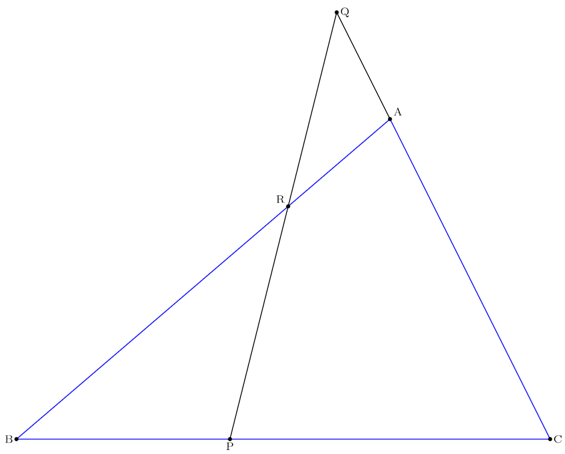


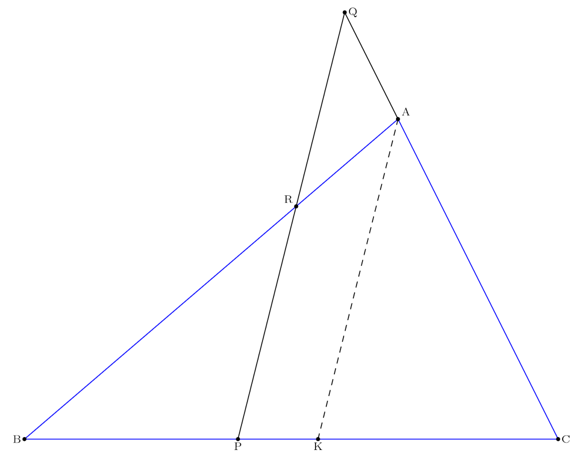


[1] http://web.evanchen.cc/handouts/Directed-Angles/Directed-Angles.pdf

The converse is also true: If points D, E, and F are chosen on BC, AC, and AB respectively so that the above equations are true, then D, E, and F are collinear. Note that the value -1 is peculiar, but it is a result of our notion of signed segments. In other words the length AB is taken to be positive or negative according to whether A is to the left or right of B in some fixed orientation of the line. For example, $\frac{AF}{FB}$ is defined as having positive value when F is between A and B and negative otherwise.

Proof: Let points P, Q, R lie on the respective sides BC, CA, AB (or their extensions). Draw a line parallel to QP through A to intersect BC at K. We have





$$\triangle RBP \sim \triangle ABK \implies \frac{AR}{RB} = \frac{KP}{PB},$$

$$\triangle QCP \sim \triangle ACK \implies \frac{QC}{QA} = \frac{PC}{PK}.$$

Multiplying the two equalities together to eliminate the PK factor, we get

$$\frac{AR}{RB} \cdot \frac{QC}{QA} = -\frac{PC}{PB} \implies \frac{AR}{RB} \cdot \frac{QC}{QA} \cdot \frac{PB}{PC} = -1. \blacksquare$$

More on Directed Segments: The two theorems above can be formulated in terms of directed segments in a more natural way. Suppose the points A, B, and C are collinear, and we include direction along with distance when defining line segments, so that if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ point in the same direction, then $AB/BC > 0$, but if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ point in the opposite direction, then $AB/BC < 0$. Now consider the quantity

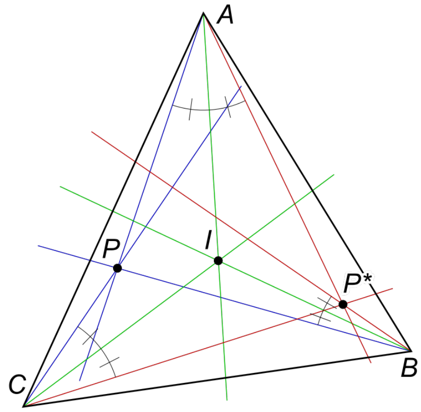
$$P' = \frac{\overline{AC_1}}{\overline{C_1B}} \cdot \frac{\overline{BA_1}}{\overline{A_1C}} \cdot \frac{\overline{CB_1}}{\overline{B_1A}}.$$

If $P' = 1$, then we have Ceva's Theorem; if $P' = -1$, then we have Menelaus' Theorem.

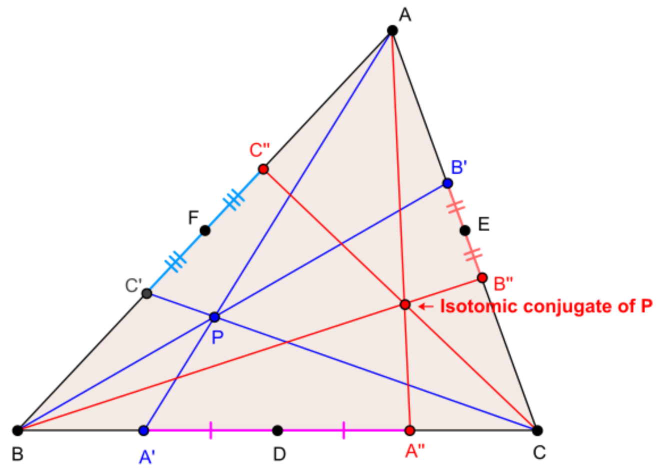
Isogonal Conjugate: The isogonal conjugate of a point P with respect to a triangle ABC

is constructed by reflecting the lines PA, PB, and PC about the angle bisectors of A, B, and C respectively. These three reflected lines concur at the isogonal conjugate of P. This definition applies only to points not on a sideline of triangle ABC. Note that the incenter I is its own isogonal conjugate.[2]

Isotomic Conjugate: Similar definition to the isogonal conjugate, only with midlines instead of angle bisectors.

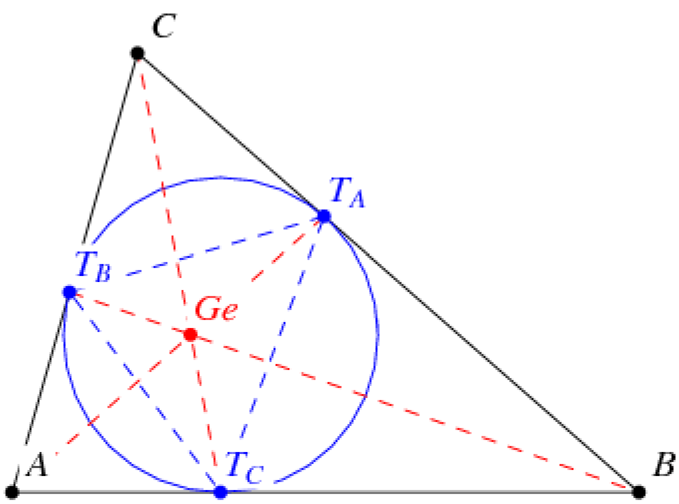


The isogonal conjugate of P.



The isotomic conjugate of P.

Gergonne Point: Let the incircle touch BC, CA, and AB at T_A, T_B, and T_C respectively. AT_A, BT_B, CT_C are concurrent at the Gergonne point of $\triangle ABC$.



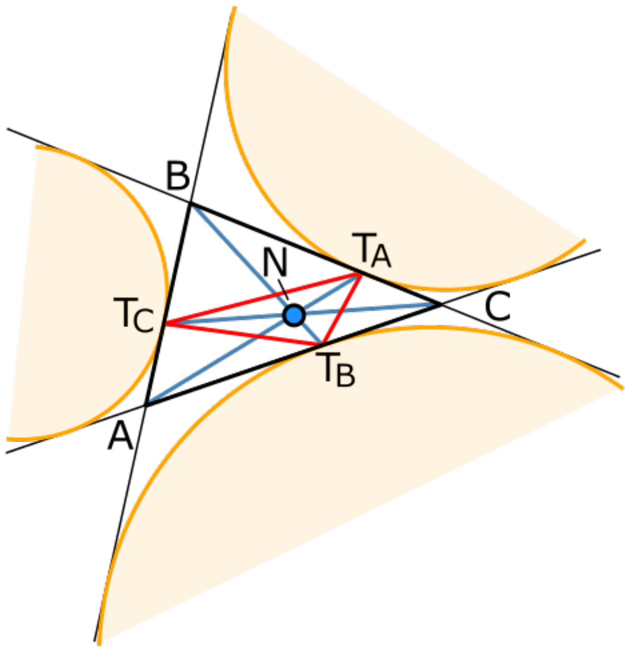
Proof: Since $AT_B = AT_C = x$, $BT_C = BT_A = y$, and $CT_A = CT_B = z$, we have

$$\frac{BT_A}{CT_A} \cdot \frac{CT_B}{AT_B} \cdot \frac{AT_C}{BTC} = \frac{y}{z} \cdot \frac{z}{x} \cdot \frac{x}{y} = 1,$$

[2] https://usamo.wordpress.com/2014/11/30/three-properties-of-isogonal-conjugates/

and the result follows from Ceva's Theorem. ■

Nagel Point: Let T_A be the point of tangency of the A excircle of ABC at BC. Similarly, T_B and T_C are defined (points of tangency of B and C excircles). Then AT_A, BT_B, and CT_C are concurrent at the Nagel point of $\triangle ABC$.

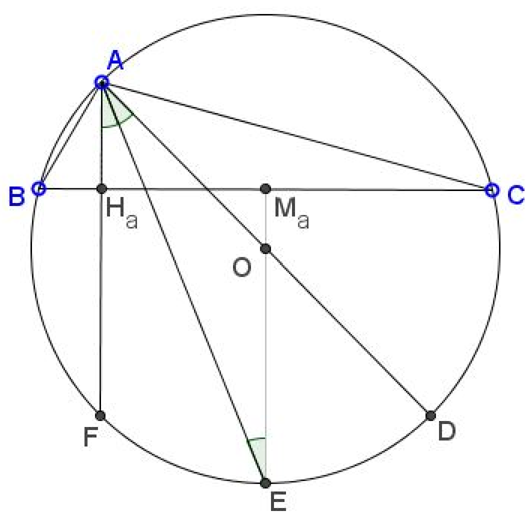


Proof: We have $AT_B = z = BT_A$, $BT_C = x = CT_B$, and $CT_A = y = AT_C$, hence

$$\frac{BT_A}{CT_A} \cdot \frac{CT_B}{AT_B} \cdot \frac{AT_B}{BT_B} = \frac{z}{y} \cdot \frac{x}{z} \cdot \frac{y}{x} = 1,$$

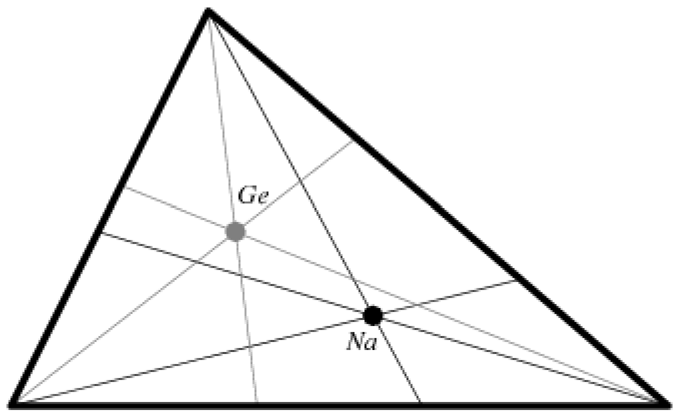
and the result follows from Ceva's Theorem. ■

Orthocenter and Circumcenter are Isogonal:



Proof: Let F be the second intersection of the altitude AH_a with circle (ABC) and and E the second intersection of the bisector. Let M_a be the midpoint of BC. Then O, M_a, and E are collinear. $\angle AEO = \angle EAO$ since $\triangle AEO$ is isosceles. Also, $\angle FAE = \angle AEO$ as alternate interior in parallel lines AF and EO. It follows that $\angle FAE = \angle EAO$. We apply this same process to all three vertices and obtain the desired result. ■

Nagel Point and Gergonne Point are Isotomic:



- Try to prove this on your own!

Nagel Line: Let I, G, and N be the incenter, the centroid, and the Nagel point of a triangle $\triangle ABC$. We have all three points collinear with radio $IG : GN = 2 : 1$.

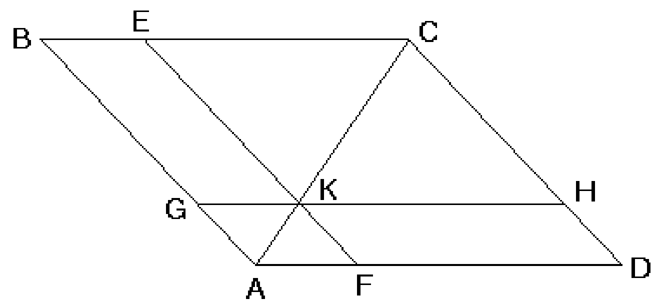
Proof: The proof of this result generally involves a technique known as homothety, which is slightly beyond the scope of this course. Another interesting point that lies on this line is the Spieker center, or the incenter of the medial triangle.[3]

Newton-Gauss Line: Let $ABCD$ be a convex quadrilateral. Lines AB and DC meet at E, AD and BC at F. Take X, Y, Z the midpoints of segments AC, BD and EF. Prove that the points X, Y, Z are collinear.

We will first provide a lemma that will help with a proof.

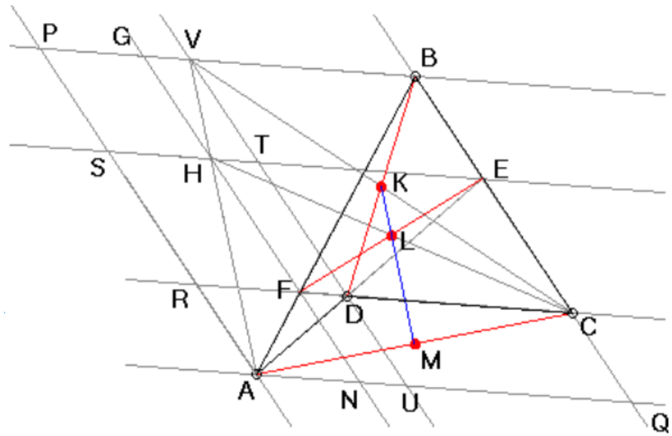
Lemma: Let $ABCD$ be a parallelogram. Lines EF and GH are parallel to the sides and meet on the diagonal AC. Then $[ABEF] = [AGHD]$ and $[GBEK] = [FKHD]$. The converse is also true.

[3] `https://artofproblemsolving.com/community/c3103h1052171_the_nagel_line`

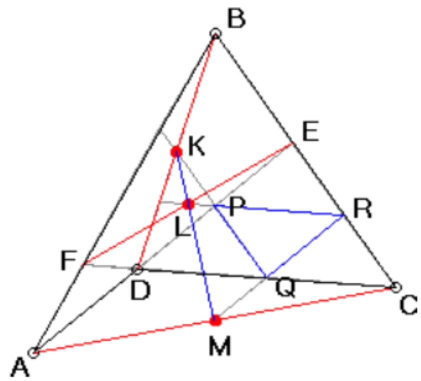


Proof of Lemma: Three pairs of triangles have equal areas: ABC and ACD, KEC and KCH, AGK and AKF. The lemma follows by comparing two algebraic sums of the areas involved. For the converse note that if parallelograms $ABEF$ and $AGHD$ have equal area while lines EF and GH do not meet on the diagonal AC we may arrive at a contradiction by drawing a line parallel to AD through the intersection of EF and AC. The new parallelogram will have the area equal to that of $ABEF$ and unequal area to that of $AGHD$. ■

We now proceed with two proofs of the Newton-Gauss line.



Proof 1: Parallelograms $ARCQ$ and $APGN$ have equal areas, and so have $ARCQ$ and $ASTU$. Therefore, the same holds for the parallelograms $PGHS$ and $HTUN$. This means that H lies on AV. Therefore, midpoints of segments CV, CH, and CA lie on a line (parallel to AV). In parallelogram $VBCD$, the midpoint of CV coincides with the midpoint of BD, also known as K. In parallelogram $FHEC$, the midpoint of CH coincides with that of FE, also known as L. Therefore, the three midpoints K, L, and M (of CA) lie on a line. ■



Proof 2: Choose $MR \parallel AE$, $KQ \parallel BC$, and $LR \parallel CD$. $MR \parallel AE$ implies $\frac{RM}{MQ} = \frac{EA}{AD}$. $KQ \parallel BC$ implies $\frac{QK}{KP} = \frac{CB}{BE}$. $LR \parallel CD$ implies $\frac{PL}{LR} = \frac{DF}{FC}$. Multiply the three proportions:

$$\frac{RM}{MQ} \cdot \frac{QK}{KP} \cdot \frac{PL}{LR} = \frac{EA}{AD} \cdot \frac{CB}{BE} \cdot \frac{DF}{FC}.$$

Consider now two triangles and their respective transversals: triangle EDC and line AFB and triangle PRQ and line KLM. The latter is yet to be shown to be a line! By Menelaus' theorem, the right hand side in the above equation equals 1. Therefore, the product on the left is also 1. By the converse of Menelaus' theorem, the points K, L, and M lie on a line. ■

Area Lemma: In $\triangle ABC$, point D lies on BC. If P is any point on cevian AD then

$$\frac{BD}{DC} = \frac{[APB]}{[APC]}.$$

Worked Examples:

1. In triangle ABC, let N be the midpoint of A-median AM and P be the point on AC such that $AP = \frac{1}{3}AC$. Prove that B, N, P are collinear.

   Solution: Menelaus Theorem in triangle AMC immediately gives

   $$\frac{AN}{NM} \cdot \frac{MB}{BC} = \frac{CP}{PA} = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{2}{1} = 1,$$

   hence result.

2. Points D, E lie on the sides BC, AC of the triangle ABC such that DE is parallel to AB. Denote the intersection of AD and BE by P. Prove that P lies on the C-median.

   Solution: Let M' be the intersection of CP and AB. By Ceva's Theorem for concurrent cevians AD, BE, CM' we have

   $$\frac{AM'}{M'B} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Since $DE \| AB$, we have $BD : DC = AE : EC$ (D and E have the same C-level) and thus $AM' = M'B$ implying that M' is the midpoint of AB. Thus, P lies on the C-median.

3. A circle meets the sides BC, CA, and AB of $\triangle ABC$ at points A_1; A_2, B_1; B_2, and C_1; C_2. Prove that the lines AA_1, BB_1, and CC_1 are concurrent if and only if the lines AA_2, BB_2, and CC_2 are concurrent.

   Solution: Recall that by Power of a Point, $AB_1 \cdot AB_2 = AC_1 \cdot AC_2$, hence $\frac{AC_1}{B_1A} = \frac{B_2A}{AC_2}$ and similarly for others. Suppose that AA_1, BB_1, and CC_1 are concurrent. Starting with Ceva's Theorem, rearranging, and applying the recalled equalities, we may write

   $$1 = \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} = \frac{AC_1}{B_1A} \cdot \frac{BA_1}{C_1B} \cdot \frac{CB_1}{A_1C} = \frac{B_2A}{AC_2} \cdot \frac{C_2B}{BA_2} \cdot \frac{A_2C}{CB_2},$$

   which by Ceva's Theorem means that AA_2, BB_2, CC_2 are concurrent. The other implication is identical.

4. Let B, D, and C be fixed points on a line such that D lies on segment BC. Let A be a variable point not on line BC. Let P be a variable point on segment AD, $BP \cap CA = E$ and $CP \cap AB = F$. Let EF intersect BC at D'. Prove that D' is a fixed point.

   Solution: Using Ceva on $\triangle ABC$ with D, E, F on BC, CA, AB, we get

   $$\frac{BF}{FA} \cdot \frac{AE}{EC} \cdot \frac{CD}{DB} = 1$$

   Using Menelaus on $\triangle ABC$ with D', E, F on BC, CA, AB, we get

   $$\frac{BF}{FA} \cdot \frac{AE}{EC} \cdot \frac{CD'}{D'B} = -1$$

   It follows that $\frac{CD}{DB} = -\frac{CD'}{D'B}$. Since B, D, C are fixed, the LHS is fixed. Thus $CD'/D'B$ is fixed. As that is a directed ratio where B, C are fixed, D' is fixed.

5. Let ABC be a triangle with $\angle A = 100°$, $\angle B = 60°$, and let $M \in BC$ and $N \in AC$ be points for which $\angle BAM = 30°$ and $\angle ABN = 20°$. Prove that the lines AM, BN and the bisector of $\angle ACB$ are concurrent.

   Solution: Denote the intersection of AM and BN by P. It suffices to prove that $\angle ACP = \angle BCP$, i.e. (using trigonometric form of Ceva's Theorem) that

   $$\frac{\sin 70°}{\sin 30°} \cdot \frac{\sin 20°}{\sin 40°} = 1,$$

which is true, since

$$\frac{\sin 70^\circ \cdot \sin 20^\circ}{\sin 30^\circ} = 2\cos 20^\circ \sin 20^\circ = \sin 40^\circ.$$

6. The diagonals AD, BE, and CF of a cyclic hexagon are concurrent iff $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

   Solution: Focus on $\triangle ACE$. Trigonometric form of Ceva's Theorem implies that AD, BE, and CF are concurrent if and only if

   $$1 = \frac{\sin \angle EAD}{\sin \angle CAD} \cdot \frac{\sin \angle ACF}{\sin \angle ECF} \cdot \frac{\sin \angle CEB}{\sin \angle AEB} = \frac{\sin \angle ECD}{\sin \angle CED} \cdot \frac{\sin \angle AEF}{\sin \angle EAF} \cdot \frac{\sin \angle CAB}{\sin \angle ACB}.$$

   The law of sines in triangles CDE, EFA, ABC implies that the latter equals

   $$\frac{DE}{CD} \cdot \frac{FA}{EF} \cdot \frac{BC}{AB}$$

   and we are done.

7. Let ABC be a triangle. Prove that lines joining midpoints of the sides with midpoints of corresponding heights pass through a common point.

   Solution: Denote the midpoints of BC, CA, AB by M, N, P and observe that the midpoints of heights lie on the sides of $\triangle MNP$, hence we want to apply Ceva in $\triangle MNP$. Denoting the midpoint of height AD by D' we see that $PD'/D'N = BD/DC$ and likewise for the others, hence the desired product rewrites as

   $$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA}$$

   (where E, F are the other feet of heights), which equals 1 by Ceva in $\triangle ABC$ for concurrent heights.

8. Let ABC be a triangle with $AB < AC$ and incenter I. On the side BC we mark the midpoint M, the point of contact of the incircle D and the foot of A-angle bisector A_1. If N is the midpoint of AD, prove that M, I, N are collinear.

   Solution: Applying the Angle Bisector Theorem twice we learn $AI/IA_1 = BA/BA_1 = (b+c)/a$. Recalling $BM = \frac{1}{2}a$, $BA_1 = \frac{ac}{b+c}$, and $BD = \frac{1}{2}(a+c-b)$, some computation yields $A_1M/MD = a/(b+c)$. Hence Menelaus in $\triangle AA_1D$ for points I, M, N finishes the proof.

9. Let the external angle bisector of $\angle BAC$ intersect BC at A'. Define B', C' analogously. Prove that A', B', C' are collinear.

   Solution: We aim to prove $\frac{A'B}{A'C} = \frac{c}{b}$ and similarly for others. Then, applying Menelaus for points A', B', C' on the respective sides of $\triangle ABC$, we learn that they are indeed collinear. One way to go about it is to observe that letting AA' horizontal, both AB and AC are equally steep. Hence indeed $\frac{A'B}{A'C} = \frac{c}{b}$, because both the sides express the ratio of heights (levels) of the points B, C above line AA'.Another way is to compute $\angle A'AB = 90° - \frac{1}{2}A$ and $\angle AA'C = 90° + \frac{1}{2}A$, note $\sin\angle A'AB = \sin\angle A'AC$, and apply Ratio Lemma.

10. Triangle ABC is acute with altitude CD (D on AB). Point P is on segment CD. Rays AP and BP are extended to hit BC and CA to Q and R respectively. Prove that $\angle QDC = \angle RDC$.

    Solution: Ceva's Theorem in $\triangle ABC$ implies

    $$\frac{CQ}{QB} \cdot \frac{BD}{DA} \cdot \frac{AR}{RC} = 1.$$

    Now, let lines DQ, DR intersect line parallel to AB and passing through C at X, Y, respectively. Then $\triangle QCX \sim \triangle QBD$ and $\triangle RCY \sim \triangle RAD$ (both by AA), so we may rewrite the first fraction as CX/BD and the third one as DA/CY. Hence the Ceva reads as

    $$\frac{CX}{BD} \cdot \frac{BD}{DA} \cdot \frac{DA}{CY} = 1 \quad \Rightarrow \quad CX = CY.$$

    In triangle DXY, line DC is both its altitude and median, so the triangle is isosceles and DC is also its angle bisector.

11. **IMO 1996 Shortlist #G4** Let P be a point inside an equilateral triangle ABC. Let the lines AP, BP, CP meet the sides BC, CA, AB at A_1, B_1, C_1. Prove that

    $$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A.$$

    Solution: Squaring and using Ceva in $\triangle ABC$, it suffices to prove $B_1C_1^2 \geq AB_1 \cdot AC_1$ (and likewise for the others) which follows from the Law of Cosines in $\triangle AB_1C_1$ and inequality $x^2 + y^2 - xy \geq xy$.