Geometry Day 2 Problem Set Solutions

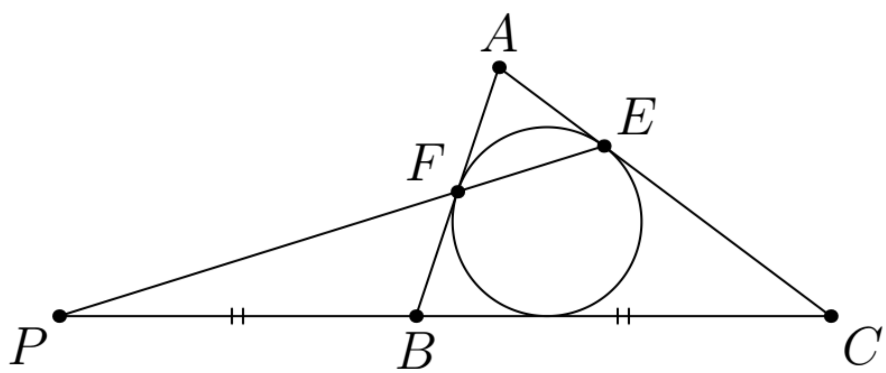
Math Circle Competition Team

October 15th, 2017

1. In $\triangle ABC$, points D, E, and F lie on BC, AC, and AB respectively such that AD, BE, and CF are concurrent. Suppose AB, AC, and BC have lengths 13, 14, and 15, respectively. If $\frac{AF}{FB} = \frac{2}{5}$ and $\frac{CE}{EA} = \frac{5}{8}$, find BD and DC.

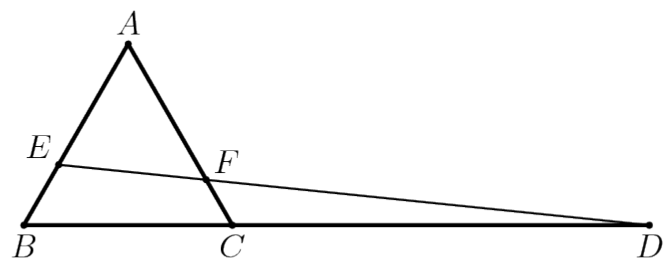
   Solution: If $BD = x$ and $DC = y$, then $10x = 40y$ by Ceva's, and $x + y = 15$. From this, we find $x = \boxed{12}$ and $y = \boxed{3}$.

2. **djmathman** Triangle ABC has $AB = 2007$ and $AC = 2015$. The incircle ω of the triangle is tangent to AC and AB at E and F respectively, and P is the intersection point of EF and BC. Suppose B is the midpoint of $\overline{CP}$. Compute the length BC.



   P, F, E are the three collinear points and we are given essentially the ratio $\frac{CP}{PB}$. We are given $c = 2007$, $b = 2015$, and we want to find a. We have $AF = AE = s - a$, $FB = s - b$, and $EC = s - c$. Therefore, by Menelauss Theorem, $\frac{CP}{PB} \cdot \frac{BF}{FA} \cdot \frac{AE}{EC} = 1 \implies 2 \cdot \frac{s-b}{s-a} \cdot \frac{s-a}{s-c} = 1$. This simplifies to $2(s - b) = s - c$. Then $2s - 2b = s - c$, so $s = 2b - c$. Hence $\frac{a+b+c}{2} = 2b - c$, and solving for a, we get $a = 3b - 3c$. With $c = 2007$ and $b = 2015$, we find that $a = 3(2015 - 2007) = \boxed{24}$.

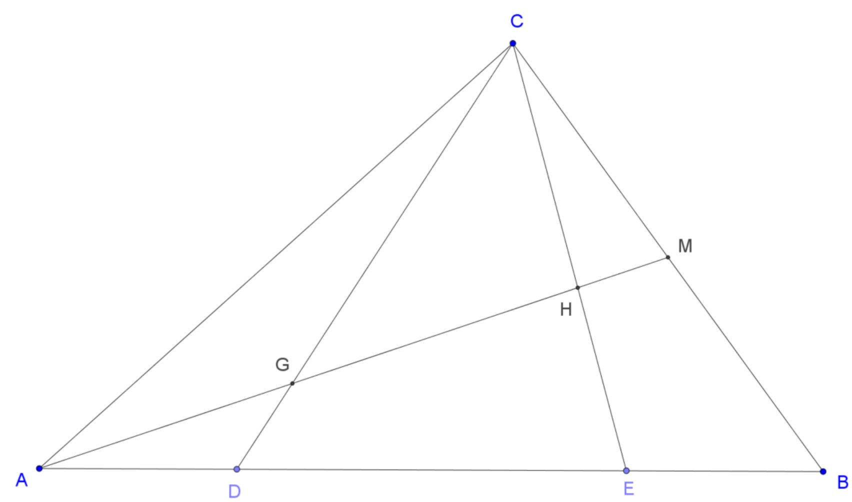
3. **Purple Comet 2014 #25** The diagram below shows equilateral $\triangle ABC$ with side length 2. Point D lies on ray $\overrightarrow{BC}$ so that $CD = 4$. Points E and F lie on $\overline{AB}$ and $\overline{AC}$, respectively, so that E, F, and D are collinear, and the area of $\triangle AEF$ is half of the area of $\triangle ABC$. Find $\frac{AE}{AF}$.



Solution: Let $AE = x$ and $AF = y$. Then, by Menelaus, we get $\frac{x}{2-x} \cdot \frac{6}{4} \cdot \frac{2-y}{y} = 1$. Furthermore we have $\frac{x}{2} \cdot \frac{y}{2} = \frac{1}{2} \implies xy = 2$ by the area condition.

As such we get $\frac{2x-xy}{2y-xy} = \frac{2}{3} \implies 6x - 3xy = 4y - 2xy \implies xy + 4y - 6x = 0$. Let $y = \frac{2}{x}$ to get $2 + \frac{8}{x} - 6x = 0 \implies 6x^2 - 2x - 8 = 0 \implies 3x^2 - x - 4 = 0 \implies x = \frac{4}{3}$.Thus, $y = \frac{3}{2}$ and $\frac{x}{y} = \boxed{\frac{8}{9}}$.

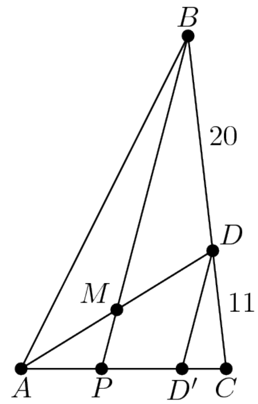
4. In triangle ABC, points D and E lie on side AB dividing the side in a ratio of 1:2:1; in other words, DE is twice as long as AD and EB. Let AM be the median to BC, and let G and H denote the intersection points of this with CD and CE, respectively. What is the ratio $AG : GH : HM$?



Solution: We will use Menelaus twice on $\triangle AMB$: once with transversal EC, and once with DC. Menelaus with EC yields $\dfrac{BC}{MC} \cdot \dfrac{HM}{AH} \cdot \dfrac{AE}{EB} = 1$. But $BM = MC$, so $\dfrac{BC}{MC} = \dfrac{2}{1}$, and using the given ratios we also have $\dfrac{AE}{EB} = \dfrac{3}{1}$, so that our first equation becomes $2 \cdot \dfrac{HM}{AH} \cdot 3 = 1$ or $6HM = AH$. Menelaus with DC yields $\dfrac{BC}{MC} \cdot \dfrac{GM}{GA} \cdot \dfrac{AD}{DB} = 1$, where $\dfrac{BC}{MC} = \dfrac{2}{1}$ and $\dfrac{AD}{DB} = \dfrac{1}{3}$. Thus $\dfrac{BC}{MC} \cdot \dfrac{GM}{GA} \cdot \dfrac{AD}{DB} = 2 \cdot \dfrac{GM}{GA} \cdot \dfrac{1}{3} = 1$, so that $\dfrac{GM}{GA} = \dfrac{3}{2}$ and $3GA = 2GM$. Now let $AG = x$, $GH = y$, and $HM = z$. We have that

$3x = 2(y+z)$, so $x = \frac{2}{3}(y+z)$, and $6z = x+y$. Letting $z = 1$ (since we only require the ratio of these terms), we have $6 = \frac{2}{3}(y+1)+y$, and $y = \frac{16}{5}$. Then $x = \frac{2}{3}(\frac{16}{5}+1) = \frac{14}{5}$, and our desired ratio is $\frac{14}{5} : \frac{16}{5} : 1$ or $\boxed{14 : 16 : 5}$.

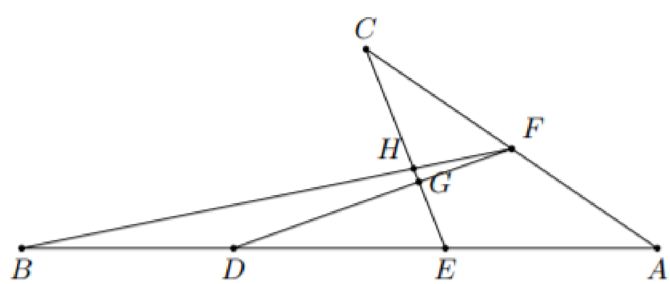
5. **AIME II 2011 #4** In triangle ABC, $AB = \frac{20}{11}AC$. The angle bisector of $\angle A$ intersects BC at point D, and point M is the midpoint of AD. Let P be the point of the intersection of AC and BM. Find the ratio of CP to PA.



Solution: By Menelaus' Theorem on $\triangle ACD$ with transversal PB,

$$1 = \frac{CP}{PA} \cdot \frac{AM}{MD} \cdot \frac{DB}{CB} = \frac{CP}{PA} \cdot \left(\frac{1}{1+\frac{AC}{AB}}\right) \quad \Longrightarrow \quad \frac{CP}{PA} = \boxed{\frac{31}{20}}.$$

6. **ARML 2012** Given noncollinear points A, B, C, segment AB is trisected by points D and E, and F is the midpoint of segment AC. DF and BF intersect CE at G and H, respectively. If $[EDG] = 18$, compute $[FGH]$.



Solution: $\frac{CG}{EG} = \frac{[CDF]}{[EDF]}$ by the area lemma. $[CDF] = \frac{1}{2}[CDA]$ and $[EDF] = \frac{1}{2}[ADF] = \frac{1}{4}[ADC]$. Thus, G is $\frac{2}{3}$ of the way from C to E. Apply area lemma again to obtain $\frac{CH}{EH} = \frac{[CBF]}{[EBF]}$. We have $[CBF] = \frac{1}{2}[CBA]$ and $[EBF] = \frac{2}{3}[ABF] = \frac{1}{3}[ABC]$. Thus, H is $\frac{3}{5}$ of the way from C to E. Now $GH = (\frac{2}{3} - \frac{3}{5})EC = \frac{1}{15}EC$. We have $[EDG] = \frac{1}{3}[EDC] = \frac{1}{9}[ABC]$ and $[FGH] = \frac{1}{15}[FEC] = \frac{1}{30}[AEC] = \frac{1}{90}[ABC]$ which is $\frac{1}{10}[EDG] = \frac{18}{10} = \boxed{\frac{9}{5}}$.