Number Theory Problem Set

Math Circle Competition Team

October 22nd, 2017

1. Compute the following:
   (a) $51 \pmod{13}$
   (b) $-463 \pmod{11}$
   (c) $7^{13} \pmod{11}$
   (d) $2^{1000} \pmod{13}$
   (e) $(1440983213234)^{123321} \pmod{5}$
2. **(Mandelbrot)** Find the last three digits of 9^{105}.
3. Find the units digit of $7^{(7^7)}$.
4. Solve the congruence $1232x \equiv 9045 \pmod{24}$
5. **(2017 AMC 10B #14)** An integer N is selected at random in the range $1 \leq N \leq 2020$. What is the probability that the remainder when N^{16} is divided by 5 is 1?
6. **(2006 AIME I #3)** Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.
7. **(2010 AIME I #2)** Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}}$ is divided by 1000.
8. **(2016 AMC 10B #25)** Let $f(x) = \sum_{k=2}^{10}(\lfloor kx \rfloor - k\lfloor x \rfloor)$, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r. How many distinct values does $f(x)$ assume for $x \geq 0$?
9. Calculate $\varphi(12)$.
10. Find $\varphi(p^k)$ for p prime and any positive integer k.
11. Prove that φ is **multiplicative**, that is, $\varphi(mn) = \varphi(m)\varphi(n)$ if m and n are relatively prime.
12. From the results of the previous two problems, find an explicit formula for $\varphi(n)$, n being any positive integer.
13. **(1989 AIME #9)** Find a positive integer n such that
$$133^5 + 110^5 + 84^5 + 27^5 = n^5.$$

14. **(2015 AIME #3)** There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p.

15. **(1983 AIME #6)** Find

$$(6^{83} + 8^{83}) \mod 49.$$