Number Theory Problem Set Solutions

Math Circle Competition Team

October 22nd, 2017

1. Compute the following:

   (a) $51 \pmod{13}$
   $\boxed{12}$ $51 = 13 \cdot 3 + 12 \equiv \boxed{12} \pmod{13}$

   (b) $-463 \pmod{11}$
   $\boxed{10}$ $-463 = 11 \cdot -43 + 10 \equiv \boxed{10} \pmod{11}$

   (c) $7^{13} \pmod{11}$
   $\boxed{2}$ By FLT, we have $7^{13} \pmod{11} = (7^{11})(7^2) \pmod{11} \equiv (7)(7^2) \pmod{11}$
   $\equiv 343 \pmod{11} \equiv \boxed{2} \pmod{11}$.

   (d) $2^{1000} \pmod{13}$
   $\boxed{3}$ By repeated FLT, we have $2^{1000} \pmod{13} = 2^{13\cdot 76+12} \pmod{13}$
   $= (2^{13})^{76}(2^{12}) \pmod{13} \equiv (2^{76})(2^{12}) \pmod{13} = 2^{88} \pmod{13}$
   $= (2^{13})^{6}(2^{10}) \pmod{13} \equiv (2^{6})(2^{10}) \pmod{13} = 2^{16} \pmod{13} = (2^{13})(2^{3}) \pmod{13}$
   $\equiv (2)(2^3) \pmod{13} = 16 \pmod{13} \equiv \boxed{3} \pmod{13}$.

   (e) $(1440983213234)^{123321} \pmod{5}$
   $\boxed{4}$ First notice that since 1440983213234 ends in a 4, it is equivalent to $-1 \mod 5$. But -1 to any odd power is -1, so that

   $$(1440983213234)^{123321} \mod 5 \equiv (-1)^{123321} \mod 5 \equiv -1 \mod 5 \equiv 4 \mod 5.$$

   Thus our answer is $\boxed{4}$.

2. **(Mandelbrot)** Find the last three digits of 9^{105}.
   $\boxed{049}$ This is equivalent to finding $9^{105} \mod 1000$. We will use the binomial theorem to simplify this. We have

   $$\begin{aligned} 9^{105} &= (10-1)^{105} \mod 11 \\ &\equiv \binom{105}{0} \cdot 10^0 \cdot (-1)^{105} + \binom{105}{1} \cdot 10^1 \cdot (-1)^{104} + \binom{105}{2} \cdot 10^2 \cdot (-1)^{103} \\ &\quad + \binom{105}{3} 10^3 \cdot (-1)^{102} + \dots \mod 1000. \end{aligned}$$

   Notice that the 10^3 term, and every term following it, will be congruent to $0 \mod 1000$.

This yields

$$\begin{aligned}
9^{105} &\equiv 1 \cdot 1 \cdot (-1) + 105 \cdot 10 \cdot 1 + \frac{105 \cdot 104}{2} \cdot 100 \cdot (-1) \mod 1000 \\
&\equiv -1 + 1050 - 105 \cdot 104 \cdot 50 \mod 1000 \\
&\equiv 1049 - 1000 \cdot 21 \cdot 26 \mod 1000 \\
&\equiv 49.
\end{aligned}$$

Thus our answer is $\boxed{049}$.

3. Find the units digit of $7^{(7^7)}$.
$\boxed{3}$ Similar to the last problem, we only need to compute the exponent mod 4 to find the units digit. Thus we have $7^7 \pmod 4 \equiv (-1)^7 \pmod 4 \equiv -1 \pmod 4 \equiv 3 \pmod 4$. So $7^{(7^7)}$ has the same units digit as $7^3 = 343$, so the answer is $\boxed{3}$.

4. Solve the congruence $1232x \equiv 9045 \pmod{24}$.
$\boxed{\text{No solutions}}$ Since x is an integer, $1232x$ will always be even, and thus $1232x \mod 24$ will also be even. But $9045 \mod 24$ must be odd, so that there are no solutions.

5. **(2017 AMC 10B #14)** An integer N is selected at random in the range $1 \leq N \leq 2020$. What is the probability that the remainder when N^{16} is divided by 5 is 1?
$\boxed{\frac{4}{5}}$ We first note that $\varphi(5) = 4$ (where φ is Euler's totient function) and use Euler's generalization of Fermat's Little Theorem. This yields $N^{16} = (N^4)^4 \equiv 1^4 \equiv 1 \mod 5$ when N is relatively prime to 5. Since only numbers ending in 5 or 0 are not relatively prime to 5, this happens with probability $\boxed{\frac{4}{5}}$.

6. **(2006 AIME I #3)** Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.
$\boxed{725}$ Suppose the original number is $N = \overline{a_n a_{n-1} \ldots a_1 a_0}$, where the a_i are digits and the first digit, a_n, is nonzero. Then the number we create is $N_0 = \overline{a_{n-1} \ldots a_1 a_0}$, so

$$N = 29N_0.$$

But N is N_0 with the digit a_n added to the left, so $N = N_0 + a_n \cdot 10^n$. Thus,

$$N_0 + a_n \cdot 10^n = 29N_0$$

$$a_n \cdot 10^n = 28N_0.$$

The right-hand side of this equation is divisible by seven, so the left-hand side must also be divisible by seven. The number 10^n is never divisible by 7, so a_n must be divisible by 7. But a_n is a nonzero digit, so the only possibility is $a_n = 7$. This gives

$$7 \cdot 10^n = 28N_0$$

or

$$10^n = 4N_0.$$

Now, we want to minimize both n and N_0, so we take $N_0 = 25$ and $n = 2$. Then

$$N = 7 \cdot 10^2 + 25 = \boxed{725}.$$

7. **(2010 AIME I)** Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}}$ is divided by 1000.
$\boxed{109}$ We wish to find $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}} \pmod{1000}$. We have $9 \cdot 99 \pmod{1000} \equiv 891 \pmod{1000}$, and $9 \cdot 99 \cdot 999 \pmod{1000} \equiv 891 \cdot (-1) \pmod{1000} \equiv 109 \pmod{1000}$. Since 9999, 99999, etc are all equivalent to $(-1) \pmod{1000}$, we have $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}} \pmod{1000} \equiv 891 \cdot (-1)^{997} \pmod{1000} \equiv 891 \cdot (-1) \pmod{1000} \equiv \boxed{109} \pmod{1000}$.

8. **(2016 AMC 10B #25)** Let $f(x) = \sum_{k=2}^{10}(\lfloor kx \rfloor - k\lfloor x \rfloor)$, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r. How many distinct values does $f(x)$ assume for $x \geq 0$?
$\boxed{32}$ Since $x = \lfloor x \rfloor + \{x\}$, we have

$$f(x) = \sum_{k=2}^{10}(\lfloor k\lfloor x \rfloor + k\{x\} \rfloor - k\lfloor x \rfloor)$$

The function can then be simplified into

$$f(x) = \sum_{k=2}^{10}(k\lfloor x \rfloor + \lfloor k\{x\} \rfloor - k\lfloor x \rfloor)$$

which becomes

$$f(x) = \sum_{k=2}^{10}\lfloor k\{x\} \rfloor$$

For each value of k, $\lfloor k\{x\} \rfloor$ can equal integers from 0 to $k-1$. The value of $\lfloor k\{x\} \rfloor$ changes only when x is equal to any of the fractions $\frac{1}{k}, \frac{2}{k} \ldots \frac{k-1}{k}$. Thus we want to count how many distinct fractions have the form $\frac{m}{n}$ where $n \leq 10$. We can find this by computing

$$\sum_{k=2}^{10}\phi(k)$$

where $\phi(k)$ is the Euler totient function. Basically, $\phi(k)$ counts the number of fractions with k as its denominator (after simplification). This comes out to be 31. Because the value of $f(x)$ is at least 0 and can increase 31 times, there are a total of $\boxed{32}$ different possible values of $f(x)$.

9. Calculate $\varphi(12)$.
$\boxed{4}$ There are 4 numbers less than 12 which are relatively prime to 12; they are 1, 5, 7, and 11 (two numbers are relatively prime if their greatest common divisor is 1). Thus $\varphi(12) = \boxed{4}$.

10. Find $\varphi(p^k)$ for p prime and any positive integer k.
$\boxed{p^k - p^{k-1}}$

11. Prove that φ is **multiplicative**, that is, $\varphi(mn) = \varphi(m)\varphi(n)$ if m and n are relatively prime.

12. From the results of the previous two problems, find an explicit formula for $\varphi(n)$, n being any positive integer.
$$\boxed{\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right)}$$

13. **(1989 AIME #9)** Find a positive integer n such that
$$133^5 + 110^5 + 84^5 + 27^5 = n^5.$$
$\boxed{144}$ Note that n is even, since the left-hand side consists of two odd and two even numbers. By Fermat's Little Theorem, we know n^5 is congruent to n modulo 5. Hence,
$$3 + 0 + 4 + 7 \equiv n \pmod 5$$
$$4 \equiv n \pmod 5.$$
Continuing, we examine the equation modulo 3 to receive
$$1 - 1 + 0 + 0 \equiv n \pmod 3$$
$$0 \equiv n \pmod 3.$$
Thus, n is divisible by three and leaves a remainder of four when divided by 5. This means that $n > 133$, so the only possibilities are $n = 144$ or $n \geq 174$. Since $n = 174$ is too large, n must be $\boxed{144}$.

14. **(2015 AIME #3)** There is a prime number p such that $16p + 1$ is the cube of a positive integer. Find p.
$\boxed{307}$ We seek an x so that $x^3 = 16p + 1$. First, since $16p + 1$ is odd, x must be odd. Next we have $x^3 - 1 = 16p$, so that $(x - 1)(x^2 + x + 1) = 16p$. Since x is odd, we must have that $(x - 1)$ is even and $(x^2 + x + 1)$ is odd. Then 16 divides $(x - 1)$, but if $(x - 1)$ were any larger multiple of 16, then p would not be prime. Thus $x - 1 = 16$ and $x = 17$. Now we just solve for p: $16(17^2 + 17 + 1) = 16p$ so that $p = \boxed{307}$.

15. **(1983 AIME #6)** Find

$$(6^{83} + 8^{83}) \mod 49.$$

$\boxed{35}$ Since $\phi(49) = 42$ (the Euler's totient function), by Euler's Totient Theorem,

$$a^{42} \equiv 1 \pmod{49}$$

where $\gcd(a, 49) = 1$. Thus $6^{83}+8^{83} \equiv 6^{2(42)-1}+8^{2(42)-1} \equiv 6^{-1}+8^{-1} \equiv \frac{8+6}{48} \equiv \frac{14}{-1} \equiv \boxed{35} \pmod{49}$.